Tracts in Mathematics 11

Hans Triebel

Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration

European Mathematical Society

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2010 Mathematical Subject Classification: 46-02, 46E35, 42C40, 42B35, 68Q17, 41A55

Key words: Function spaces, Haar bases, Faber bases, sampling, numerical integration, discrepancy

ISBN 978-3-03719-085-2

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the author's TEX files: I. Zimmermann, Freiburg
Printed in Germany

9 8 7 6 5 4 3 2 1

Preface

Although this book deals with some selected topics of the theory of function spaces and the indicated applications, we tried to make it independently readable. For this purpose we provide in Chapter 1 notation and basic facts, give detailed references and prove some specific assertions.

Chapters 2 and 3 deal with Haar bases and Faber bases in function spaces of type B^s_{pq} and F^s_{pq}, covering some (fractional) Sobolev spaces, (classical) Besov spaces and Hölder–Zygmund spaces. In higher dimensions preference is given to several types of spaces with dominating mixed smoothness. This paves the way to study in Chapters 4 and 5 sampling and numerical integration for corresponding spaces on cubes and more general domains. It is well known that numerical integration is symbiotically related to discrepancy, the theory of irregularities of distribution of points, preferably in cubes. This is subject of Chapter 6.

Formulas are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. Chapter n is divided in sections $n.k$ and subsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Section $n.k$ or Subsection $n.k.l$. If there is no danger of confusion (which is mostly the case) we write A^s_{pq}, $S^r_{pq}A$, ..., a^s_{pq}, $s^r_{pq}a$ ...(spaces) instead of $A^s_{p,q}$, $S^r_{p,q}A$, ..., $a^s_{p,q}$, $s^r_{p,q}a$ .... Similarly a_{jm}, λ_{jm}, Q_{km} (functions, numbers, rectangles) instead of $a_{j,m}$, $\lambda_{j,m}$, $Q_{k,m}$ etc. References ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations. The number(s) behind ► in the Bibliography mark the page(s) where the corresponding entry is quoted. log is always taken to base 2. All unimportant positive constants will be denoted by c (with additional marks if there are several c's in the same formula). Our use of $\sim$ (equivalence) is explained on p. 176.

It is a pleasure to acknowledge the great help I have received from my colleagues and friends who made valuable suggestions which have been incorporated in the text. I am especially indebted to Dorothee D. Haroske for her remarks and for producing all the figures and to Erich Novak for many stimulating discussions.

Jena, Spring 2010 Hans Triebel

Contents

Chapter 1
Function spaces

1.1 Isotropic spaces

1.1.1 Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} \tag{1.1}$$

with the natural modification if $p = \infty$. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \sum_{j=1}^{n} \alpha_j. \tag{1.2}$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{(monomials)}. \tag{1.3}$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \,\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (1.4) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \leq 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \geq 3/2, \tag{1.5}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \ k \in \mathbb{N}. \tag{1.6}$$

Since

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

the φ_j form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \; s \in \mathbb{R}. \tag{1.8}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee |L_p(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty \tag{1.9}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \; s \in \mathbb{R}. \tag{1.10}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^\infty 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.11}$$

(with the usual modification if $q = \infty$).

Remark 1.2. The theory of these spaces and their history may be found in [T83], [T92], [T06]. In particular these spaces are independent of admitted resolutions of unity φ according to (1.5)–(1.7) (equivalent quasi-norms). This justifies our omission of the subscript φ in (1.9), (1.11) in the sequel. We remind the reader of a few special cases and properties referring for details to the above books, especially Section 1.2 in [T06].

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{1.12}$$

is a well-known *Littlewood–Paley theorem*.

(ii) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \tag{1.13}$$

are the *classical Sobolev spaces* usually equivalently normed by

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|^p \Big)^{1/p}. \tag{1.14}$$

This generalises (1.12).

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma : f \mapsto \big(\langle \xi \rangle^\sigma \hat{f} \big)^\vee \quad \text{with } \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \tag{1.15}$$

is a one-to-one map of $S(\mathbb{R}^n)$ onto itself and of $S'(\mathbb{R}^n)$ onto itself. Furthermore, I_σ is a lift for the spaces $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ and $s \in \mathbb{R}$, $0 < p \leq \infty$ ($p < \infty$ for the F-scale), $0 < q \leq \infty$,

$$I_\sigma A^s_{pq}(\mathbb{R}^n) = A^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.16}$$

(equivalent quasi-norms). With

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.17}$$

one has

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \qquad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.18}$$

and

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad k \in \mathbb{N}_0,\ 1 < p < \infty. \tag{1.19}$$

Nowadays one calls $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes fractional Sobolev spaces or Bessel-potential spaces) with the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ as special cases.

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{1.20}$$

as *Hölder–Zygmund spaces*. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h(\Delta^l_h f)(x), \tag{1.21}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\| f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup |h|^{-s} |\Delta^m_h f(x)| \tag{1.22}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \leq 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$.

(v) This assertion can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. Then

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \| f \,|L_p(\mathbb{R}^n)\| + \Big(\int_{|h|\leq 1} |h|^{-sq} \, \| \Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}h}{|h|^n} \Big)^{1/q} \tag{1.23}$$

and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|^*_m = \| f \,|L_p(\mathbb{R}^n)\| + \Big(\int_0^1 t^{-sq} \sup_{|h|\leq t} \| \Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{dt}{t} \Big)^{1/q} \tag{1.24}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. These are the *classical Besov spaces*. We refer to [T92, Chapter 1] and [T06, Chapter 1], where one finds the history of these spaces, further special cases and classical assertions. In addition, (1.23), (1.24) remain to be equivalent quasi-norms in

$$B^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < p, q \leq \infty \ \text{ and } \ s > n\big(\max\big(\tfrac{1}{p}, 1\big) - 1\big), \tag{1.25}$$

[T83, Theorem 2.5.12, p. 110].

1.1.2 Atoms

We give a detailed description of atomic representations of the spaces introduced in Definition 1.1 adapted to our later needs.

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm} and

$$\chi_{jm}^{(p)}(x) = 2^{jn/p}\chi_{jm}(x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.26}$$

its p-normalised modification where $0 < p \le \infty$.

Definition 1.3. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\} \tag{1.27}$$

such that

$$\|\lambda\,|b_{pq}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty \tag{1.28}$$

and f_{pq} is the collection of all sequences λ according to (1.27) such that

$$\|\lambda\,|f_{pq}\| = \Big\|\Big(\sum_{j,m}|\lambda_{jm}\chi_{jm}^{(p)}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.29}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.4. One has $b_{pp} = f_{pp}$, $0 < p \le \infty$.

Definition 1.5. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d > 1$. Then the L_∞-functions $a_{jm}: \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0, m \in \mathbb{Z}^n$ are called *(s,p)-atoms* if

$$\operatorname{supp} a_{jm} \subset d\,Q_{jm}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n; \tag{1.30}$$

there exist all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \le K-1$ such that

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.31}$$

$$|D^\alpha a_{jm}(x) - D^\alpha a_{jm}(y)| \le 2^{-j(s-\frac{n}{p})+jK}|x-y|, \quad |\alpha| = K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.32}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < L,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{1.33}$$

Remark 1.6. No cancellation (1.33) for $a_{0,m}$ is required. Furthermore, if $L = 0$ then (1.33) is empty (no condition). If $K = 0$ then (1.31), (1.32) means that $a_{jm} \in L_\infty(\mathbb{R}^n)$ and $|a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})}$. Of course, the conditions for the above atoms depend not only on s and p, but also on the given numbers K, L, d. But this will only be indicated when extra clarity is required and denoted as $(s,p)_{K,L,d}$-atoms. Otherwise we speak about (s,p)-atoms. If one replaces (1.31), (1.32) by

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K, \; j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n, \tag{1.34}$$

then the above definition coincides essentially with [T08, Definition 1.5]. But in connection with spline wavelets it is convenient to replace (1.34) with $|\alpha| = K$ by (1.32) assuming that the derivatives of order $K-1$ are only subject to the indicated Lipschitz conditions. This is immaterial for the following atomic representation theorem. Let as usual

$$\sigma_p = n\Big(\max\Big(\frac{1}{p},1\Big)-1\Big), \quad \sigma_{pq} = n\Big(\max\Big(\frac{1}{p},\frac{1}{q},1\Big)-1\Big). \tag{1.35}$$

Theorem 1.7. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_p - s, \tag{1.36}$$

and $d > 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{1.37}$$

where a_{jm} *are* (s,p)*-atoms (more precisely* $(s,p)_{K,L,d}$*-atoms) according to Definition* 1.5 *and* $\lambda \in b_{pq}$. *Furthermore,*

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{1.38}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.37) *(for fixed* K, L, d*).*

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_{pq} - s, \tag{1.39}$$

and $d > 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (1.37) *where* a_{jm} *are* (s,p)*-atoms (more precisely* $(s,p)_{K,L,d}$*-atoms) according to Definition* 1.5 *and* $\lambda \in f_{pq}$. *Furthermore,*

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\| \tag{1.40}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.37) *(for fixed* K, L, d*).*

Remark 1.8. Recall that dQ_{jm} are cubes centred at $2^{-j}m$ with side-length $d\,2^{-j}$ where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. For fixed d with $d > 1$ and $j \in \mathbb{N}_0$ there is some overlap of the cubes dQ_{jm} where $m \in \mathbb{Z}^n$. This makes clear that Theorem 1.7 based on Definition 1.5 is reasonable. Otherwise the above formulation with (1.34) in place of (1.31), (1.32) coincides essentially with [T06, Section 1.5.1]. There one finds also technical comments how the convergence in (1.37) must be understood. The replacement of (1.34) by (1.31), (1.32) is justified by the more general assertion in [T06, Corollary 1.23] which in turn is based on [TrW96], [ET96], but essentially also covered by [FrJ85], [FrJ90]. Otherwise we refer to [T92, Section 1.9] where we described the rather involved history of atoms in function spaces.

1.1.3 Local means

Assertions for local means are dual to atomic representations according to Theorem 1.7 as far as smoothness assumptions and cancellation properties are concerned. We rely on [T08, Section 1.13] where we developed a corresponding theory. Let Q_{jm} be the same cubes in $\mathbb{R}^n$ as in the previous Section 1.1.2.

Definition 1.9. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then the L_∞-functions $k_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called kernels (of local means) if

$$\operatorname{supp} k_{jm} \subset C\,Q_{jm}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n; \tag{1.41}$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \le A-1$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.42}$$

$$|D^\alpha k_{jm}(x) - D^\alpha k_{jm}(y)| \le 2^{jn+jA}|x-y|, \quad |\alpha| = A-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.43}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} x^\beta\, k_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < B,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{1.44}$$

Remark 1.10. No cancellation (1.44) for $k_{0,m}$ is required. Furthermore, if $B = 0$ then (1.44) is empty (no condition). If $A = 0$ then (1.42), (1.43) means $k_{jm} \in L_\infty(\mathbb{R}^n)$ and $|k_{jm}(x)| \le 2^{jn}$. Compared with Definition 1.5 for atoms we have different normalisations. We adapt the sequence spaces introduced in Definition 1.3 correspondingly. Recall that χ_{jm} are the characteristic functions of the cubes Q_{jm}.

Definition 1.11. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $\bar{b}^s_{pq}$ is the collection of all sequences λ according to (1.27) such that

$$\|\lambda\,|\bar{b}^s_{pq}\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty \tag{1.45}$$

and $\bar{f}^s_{pq}$ is the collection of all sequences λ according to (1.27) such that

$$\|\lambda \,|\bar{f}^s_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jsq}\, |\lambda_{jm}\, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.46}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.12. The notation b^s_{pq} and f^s_{pq} (without bar) will be reserved for a slight modification of the above sequence spaces in connection with wavelet representations. Similarly as in Remark 1.4 one has $\bar{b}^s_{pp} = \bar{f}^s_{pp}$, $0 < p \le \infty$.

Definition 1.13. Let $f \in B^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let k_{jm} be the kernels according to Definition 1.9 with $A > \sigma_p - s$ where σ_p is given by (1.35) and $B = 0$. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.47}$$

are *local means*, considered as dual pairing within $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$. Furthermore,

$$k(f) = \big\{k_{jm}(f) : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\big\}. \tag{1.48}$$

Remark 1.14. Let $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$ and $p' = \infty$ if $0 < p < 1$. Then

$$B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n)', \quad s \in \mathbb{R},\ 0 < p < \infty, \tag{1.49}$$

is the dual space of $B^s_{pp}(\mathbb{R}^n)$, [T83, Theorems 2.11.2, 2.11.3]. We refer also to Theorem 1.20 below. Then $A > \sigma_p - s$ justifies (1.47) as a dual pairing. We refer for details to [T08, Remark 1.14].

Theorem 1.15. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *be the kernels according to Definition* 1.9 *where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_p - s, \quad B > s, \tag{1.50}$$

and $C > 0$ *are fixed. Let* $k(f)$ *be as in* (1.47), (1.48). *Then for some* $c > 0$ *and all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\, \|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.51}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *and* $k(f)$ *be the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_{pq} - s, \quad B > s, \tag{1.52}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.53}$$

Remark 1.16. A proof of this theorem with

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.54}$$

instead of (1.42), (1.43) may be found in [T08, pp. 7–12] based on [Tri08]. This is the same type of replacement as in (1.34) compared with (1.31), (1.32). On this basis one can follow the proof of [T08, Theorem 1.15] without any changes. Then one obtains the above theorem.

1.1.4 Wavelets

In what follows we rely in addition to atoms and local means on wavelets. We collect what we need later on. We suppose that the reader is familiar with wavelets in $\mathbb{R}^n$ of Daubechies type and the related multiresolution analysis. The standard references are [Dau92], [Mal99], [Mey92], [Woj97]. A short summary of what is needed may also be found in [T06, Section 1.7]. We give first a brief description of some basic notation. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all complex-valued continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u inclusively. Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \ u \in \mathbb{N}, \tag{1.55}$$

be *real* compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v \, dx = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u. \tag{1.56}$$

Recall that ψ_F is called the *scaling function* (father wavelet) and ψ_M the associated *wavelet* (mother wavelet). We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual multiresolution procedure. Let

$$G = (G_1, \dots, G_n) \in G^0 = \{F, M\}^n, \tag{1.57}$$

which means that G_r is either F or M. Let

$$G = (G_1, \dots, G_n) \in G^j = \{F, M\}^{n*}, \quad j \in \mathbb{N}, \tag{1.58}$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^n \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, \ m \in \mathbb{Z}^n, \tag{1.59}$$

where (now) $j \in \mathbb{N}_0$. We always assume that ψ_F and ψ_M in (1.55) have L_2-norm 1. *Then*

$$\{\Psi^j_{G,m} : j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\} \tag{1.60}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (for any $u \in \mathbb{N}$) and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j \tag{1.61}$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Psi_{G,m}^j(x)\, \mathrm{d}x = 2^{jn/2}\big(f, \Psi_{G,m}^j\big) \tag{1.62}$$

is the corresponding expansion, adapted to our needs, where $2^{-jn/2}\Psi_{G,m}^j$ are uniformly bounded functions (with respect to j and m). In [T08], based on [HaT05], [Tri04], [T06], we dealt in detail with an extension of the L_2-theory to spaces of type B and F, with and without weights, on $\mathbb{R}^n$, the n-torus $\mathbb{T}^n$, smooth and rough domains and manifolds. In what follows we need only corresponding assertions for $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$. We give a brief description. First we adapt the sequence spaces introduced in Definition 1.11 to the extra summation over G in (1.60). The characteristic function χ_{jm} of Q_{jm} has the same meaning as there.

Definition 1.17. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq}^s is the collection of all sequences

$$\lambda = \big\{\lambda_m^{j,G} \in \mathbb{C} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{1.63}$$

such that

$$\|\lambda \,|b_{pq}^s\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p\Big)^{q/p}\Big)^{1/q} < \infty \tag{1.64}$$

and f_{pq}^s is the collection of all sequences λ in (1.63) such that

$$\|\lambda \,| f_{pq}^s\| = \Big\|\Big(\sum_{j,G,m} 2^{jsq} |\lambda_m^{j,G}\, \chi_{jm}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.65}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

The wavelets $\Psi_{G,m}^j$ in (1.59), (1.61) may serve as atoms according to Definition 1.5 (appropriately normalised) and $\lambda_m^{j,G}(f)$ in (1.62) as local means as introduced in Definitions 1.9, 1.13. Then one can ask under which circumstances the Theorems 1.7, 1.15 can be applied. Otherwise we use standard notation naturally extended from Banach spaces to quasi-Banach spaces. In particular, $\{b_j\}_{j=1}^{\infty} \subset B$ in a separable complex quasi-Banach space B is called a *basis* if any $b \in B$ can be uniquely represented as

$$b = \sum_{j=1}^{\infty} \lambda_j\, b_j, \quad \lambda_j \in \mathbb{C} \quad \text{(convergence in } B\text{)}. \tag{1.66}$$

A basis $\{b_j\}_{j=1}^{\infty}$ is called an *unconditional basis* if for any rearrangement σ of $\mathbb{N}$ (one-to-one map of $\mathbb{N}$ onto itself) $\{b_{\sigma(j)}\}_{j=1}^{\infty}$ is again a basis and

$$b = \sum_{j=1}^{\infty} \lambda_{\sigma(j)} \, b_{\sigma(j)} \quad \text{(convergence in } B) \tag{1.67}$$

for any $b \in B$ with (1.66). Standard bases of separable sequence spaces as considered in this book are always unconditional. A basis in a separable quasi-Banach space which is not unconditional is called a *conditional basis*. We refer to [AlK06] for details about bases in Banach (sequence) spaces. As justified at the beginning of [T06, Section 3.1.3] we abbreviate the right-hand side of (1.61) in what follows by

$$\sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, \Psi_{G,m}^j \tag{1.68}$$

since the conditions for the sequences λ always ensure that the corresponding series converges unconditionally at least in $S'(\mathbb{R}^n)$, which means that any rearrangement converges in $S'(\mathbb{R}^n)$ and has the same limit. *Local convergence* in $B^{\sigma}_{pq}(\mathbb{R}^n)$ means convergence in $B^{\sigma}_{pq}(K)$ for any ball K in $\mathbb{R}^n$. Similarly for $F^{\sigma}_{pq}(\mathbb{R}^n)$. Recall that σ_p and σ_{pq} are given by (1.35).

Theorem 1.18. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and let* $\Psi^j_{G,m}$ *be the wavelets* (1.59) *based on* (1.55), (1.56) *with*

$$u > \max(s, \sigma_p - s). \tag{1.69}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, \Psi_{G,m}^j, \quad \lambda \in b^s_{pq}, \tag{1.70}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any space* $B^{\sigma}_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation* (1.70) *is unique,*

$$\lambda_m^{j,G} = 2^{jn/2} \big(f, \Psi^j_{G,m} \big) \tag{1.71}$$

and

$$I : f \mapsto \big\{ 2^{jn/2} \big(f, \Psi^j_{G,m} \big) \big\} \tag{1.72}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ *onto* b^s_{pq}. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\big\{ \Psi^j_{G,m} \big\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.73}$$