BIRKHÄUSER

Progress in Mathematics
255

Representation Theory and Automorphic Forms

Toshiyuki Kobayashi
Wilfried Schmid
Jae-Hyun Yang
Editors

Representation Theory and Automorphic Forms

Toshiyuki Kobayashi
Wilfried Schmid
Jae-Hyun Yang
Editors

Birkhäuser
Boston • Basel • Berlin

Toshiyuki Kobayashi
RIMS, Kyoto University
Sakyo-ku, Kyoto, 606-8502
Japan
toshi@kurims.kyoto-u.ac.jp

Wilfried Schmid
Department of Mathematics
Harvard University
Cambridge, MA 02138
U.S.A.
schmid@math.harvard.edu

Jae-Hyun Yang
Department of Mathematics
Inha University
Incheon 402-751
Republic of Korea
jhyang@inha.ac.kr

Mathematics Subject Classifications (2000): 11R39, 11R42, 14G35, 22E45, 22E50, 22E55

Library of Congress Control Number: 2007933203

ISBN-13: 978-0-8176-4505-2 e-ISBN-13: 978-0-8176-4646-2

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

www.birkhauser.com (KeS/MP)

BIRKHÄUSER

Progress in Mathematics
Volume 255

Preface

Over the last half century, deep connections between representation theory and automorphic forms have been established, using a wide range of methods from algebra, geometry and analyis. In light of these developments, Changho Keem, Toshiyuki Kobayashi and Jae-Hyun Yang organized an international symposium entitled "Representation Theory and Automorphic Forms", with the hope that a broad discussion of recent ideas and techniques would lead to new breakthroughs in the field. The symposium was held at Seoul National University, Republic of Korea, February 14–17, 2005.

This volume is an outgrowth of the symposium. The lectures cover a variety of aspects of representation theory and autmorphic forms, among them, a lifting of elliptic cusp forms to Siegel and Hermitian modular forms (T. Ikeda), systematic and synthetic applications of the original theory of "visible actions" on complex manifolds to "multiplicity-free" theorems, in particular, to branching problems for reductive symmetric pairs (T. Kobayashi), an adaption of the Rankin–Selberg method to the setting of automorphic distributions (S. Miller and W. Schmid), recent developments in the Langlands functoriality conjecture and their relevance to certain conjectures in number theory, such as the Ramanujan and Selberg conjectures (F. Shahidi), cuspidality-irreducibility relation for automorphic representations (D. Ramakrishnan), and applications of Borcherds automorphic forms to the study of discriminants of certain $K3$ surfaces with involution that arise from the theory of hypergeometric functions (K.-I. Yoshikawa). By presenting some of the most active topics in the field, the editors hope that this volume will serve as an up-to-date introduction to the subject.

Acknowledgments

We thank the invited speakers for their enthusiastic lectures and the articles they have contributed. The referees also deserve our gratitude for their important role.

The symposium was initiated by, and received funding from, the Brain Korea 21 Mathematical Sciences Division of Seoul National University, BK21-MSD-SNU for

short. Our thanks go to Hyuk Kim, Director of BK21-MSD-SNU, and to Chong-Kyu Han, Chairman of Department of Mathematics at SNU, for their support. We are especially indebted to Sung-Hoon Park, President of the JEI Corporation, and to Jee Hoon Park, President of JEI Distribution Co., LTD – and coincidentally a friend of the last-named editor; without the very generous financial contribution of the JEI Corporation the symposium would have been far more modest in scale.

Jaeyeon Joo and Eun-Soon Hong, secretaries of BK21-MSN-SNU, and Dong-Soo Shin did a splendid job preparing the symposium and catering to the needs of the participants. We are grateful also to Ann Kostant and Avanti Paranjpye of Birkhäuser Boston for their work in publishing this volume.

January 2007

Toshiyuki Kobayashi
Wilfried Schmid
Jae-Hyun Yang

Contents

1

Irreducibility and Cuspidality

Dinakar Ramakrishnan

253-37 Caltech
Pasadena, CA 91125, USA
dinakar@caltech.edu

Summary. Suppose ρ is an n-dimensional representation of the absolute Galois group of $\mathbb{Q}$ which is associated, via an identity of L-functions, with an automorphic representation π of $\mathrm{GL}(n)$ of the adele ring of $\mathbb{Q}$. It is expected that π is cuspidal if and only if ρ is irreducible, though nothing much is known in either direction in dimensions > 2. The object of this article is to show for $n < 6$ that the cuspidality of a *regular algebraic* π is implied by the irreducibility of ρ. For $n < 5$, it suffices to assume that π is semi-regular.

Key words: irreducibility, Galois representations, cuspidality, automorphic representations, general linear group, symplectic group, regular algebraic representations

Subject Classifications: 11F70; 11F80; 22E55

Introduction

Irreducible representations are the building blocks of general, semisimple Galois representations ρ, and *cuspidal* representations are the building blocks of automorphic forms π of the general linear group. It is expected that when an object of the former type is associated to one of the latter type, usually in terms of an identity of L-functions, the irreducibility of the former should imply the cuspidality of the latter, and vice versa. It is not a simple matter to prove this expectation, and nothing much is known in dimensions > 2. We will start from the beginning and explain the problem below, and indicate a result (in one direction) at the end of the introduction, which summarizes what one can do at this point. The remainder of the paper will be devoted to showing how to deduce this result by a synthesis of known theorems and some new ideas. We will be concerned here only with the *so-called easier* direction of showing the cuspidality of π given the irreducibility of ρ, and refer to [Ra5] for a more difficult result going the other way, which uses crystalline representations as well as a

*Partially supported by the NSF through the grant DMS-0402044.

refinement of certain deep modularity results of Taylor, Skinner–Wiles, et al. Needless to say, *easier* does not mean *easy*, and the significance of the problem stems from the fact that it does arise (in this direction) naturally. For example, π could be a functorial, automorphic image $r(\eta)$, for η a cuspidal automorphic representation of a product of smaller general linear groups: $H(\mathbb{A}) = \prod_j GL(m_j, \mathbb{A})$, with an associated Galois representation σ such that $\rho = r(\sigma)$ is irreducible. If the automorphy of π has been established by using a flexible converse theorem ([CoPS1]), then the cuspidality of π is not automatic. In [RaS], we had to deal with this question for cohomological forms π on GL(6), with $H = \mathrm{GL}(2) \times \mathrm{GL}(3)$ and r the Kronecker product, where π is automorphic by [KSh1]. Besides, the main result (Theorem A below) of this paper implies, as a consequence, the cuspidality of $\pi = \mathrm{sym}^4(\eta)$ for η defined by any non-CM holomorphic newform φ of weight ≥ 2 relative to $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$, without appealing to the criterion of [KSh2]; here the automorphy of π is known by [K] and the irreducibility of ρ by [Ri].

Write $\overline{\mathbb{Q}}$ for the field of all algebraic numbers in $\mathbb{C}$, which is an infinite, mysterious Galois extension of $\mathbb{Q}$. One could say that the central problem in algebraic number theory is to understand this extension. *Class field theory*, one of the towering achievements of the twentieth century, helps us understand the *abelian* part of this extension, though there are still some delicate, open problems even in that well traversed situation.

Let $\mathcal{G}_{\mathbb{Q}}$ denote the absolute Galois group of $\mathbb{Q}$, meaning $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It is a profinite group, being the projective limit of finite groups $\mathrm{Gal}(K/\mathbb{Q})$, as K runs over number fields which are normal over $\mathbb{Q}$. For fixed K, the Tchebotarev density theorem asserts that every conjugacy class C in $\mathrm{Gal}(K/\mathbb{Q})$ is the *Frobenius class* for an infinite number of primes p which are unramified in K. This shows the importance of studying the *representations* of Galois groups, which are intimately tied up with conjugacy classes. Clearly, every $\mathbb{C}$-representation, i.e., a homomorphism into $\mathrm{GL}(n, \mathbb{C})$ for some n, of $\mathrm{Gal}(K/\mathbb{Q})$ pulls back, via the canonical surjection $\mathcal{G}_{\mathbb{Q}} \to \mathrm{Gal}(K/\mathbb{Q})$, to a representation of $\mathcal{G}_{\mathbb{Q}}$, which is continuous for the profinite topology.

Conversely, one can show that every *continuous* $\mathbb{C}$-representation ρ of $\mathcal{G}_{\mathbb{Q}}$ is such a pullback, for a suitable finite Galois extension $K/\mathbb{Q}$. E. Artin associated an L-function, denoted $L(s, \rho)$, to any such ρ, such that the arrow $\rho \to L(s, \rho)$ is additive and inductive. He conjectured that for any non-trivial, irreducible, continuous $\mathbb{C}$-representation ρ of $\mathcal{G}_{\mathbb{Q}}$, $L(s, \rho)$ is entire, and this conjecture is open in general. Again, one understands well the *abelian* situation, i.e., when ρ is a 1-dimensional representation; the kernel of such a ρ defines an abelian extension of $\mathbb{Q}$. By class field theory, such a ρ is associated to a character ξ of finite order of the idele class group $\mathbb{A}^*/\mathbb{Q}^*$; here, being *associated* means they have the same L-function, with $L(s, \xi)$ being the one introduced by Hecke, *albeit* in a different language. As usual, we are denoting by $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ the topological *ring of adeles*, with $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, and by $\mathbb{A}^*$ its multiplicative group of *ideles*, which can be given the structure of a locally compact abelian topological group with discrete subgroup $\mathbb{Q}^*$.

Now fix a prime number ℓ, and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers $\mathbb{Q}_\ell$, equipped with an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. Consider the set $\mathcal{R}_\ell(n, \mathbb{Q})$ of continuous, semisimple representations

$$\rho_\ell : \mathcal{G}_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell),$$

up to equivalence. The image of $\mathcal{G}_{\mathbb{Q}}$ in such a representation is usually not finite, and the simplest example of that is given by the ℓ-adic cyclotomic character χ_ℓ given by the action of $\mathcal{G}_{\mathbb{Q}}$ on all the ℓ-power roots of unity in $\overline{\mathbb{Q}}$. Another example is given by the 2-dimensional ℓ-adic representation on all the ℓ-power *division points* of an elliptic curve E over $\mathbb{Q}$.

The correct extension to the non-abelian case of the *idele class character*, which appears in class field theory, is the notion of an irreducible *automorphic representation* π of $\mathrm{GL}(n)$. Such a π is in particular a representation of the locally compact group $\mathrm{GL}(n, \mathbb{A}_F)$, which is a restricted direct product of the local groups $\mathrm{GL}(n, \mathbb{Q}_v)$, where v runs over all the primes p and ∞ (with $\mathbb{Q}_\infty = \mathbb{R}$). There is a corresponding factorization of π as a tensor product $\otimes_v \pi_v$, with all but a finite number of π_p being *unramified*, i.e., admitting a vector fixed by the maximal compact subgroup K_v. At the archimedean place ∞, π_∞ corresponds to an n-dimensional, semisimple representation $\sigma(\pi_\infty)$ of the real Weil group $W_{\mathbb{R}}$, which is a non-trivial extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^*$. Globally, by Schur's lemma, the center $Z(\mathbb{A}) \simeq \mathbb{A}^*$ acts by a quasi-character ω, which must be trivial on $\mathbb{Q}^*$ by the automorphy of π, and so defines an idele class character. Let us restrict to the central case when π is essentially unitary. Then there is a (unique) real number t such that the twisted representation $\pi_u := \pi(t) = \pi \otimes |\cdot|^t$ is unitary (with unitary central character ω_u). We are, by abuse of notation, writing $|\cdot|^t$ to denote the quasi-character $|\cdot|^t \circ \det$ of $\mathrm{GL}(n, \mathbb{A})$, where $|\cdot|$ signifies the adelic absolute value, which is trivial on $\mathbb{Q}^*$ by the Artin product formula.

Roughly speaking, to say that π is automorphic means π_u appears (in a weak sense) in $L^2(Z(\mathbb{A})\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A}), \omega_u)$, on which $\mathrm{GL}(n, \mathbb{A}_F)$ acts by right translations. A function φ in this L^2-space whose averages over all the horocycles are zero is called a *cusp form*, and π is called *cuspidal* if π_u is generated by the right $\mathrm{GL}(n, \mathbb{A}_F)$-translates of such a φ. Among the automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ are certain distinguished ones called *isobaric automorphic representations*. Any isobaric π is of the form $\pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$, where each π_j is a cuspidal representation of $\mathrm{GL}(n_j, \mathbb{A})$, such that $(n_1, n_2, \ldots, n_r)$ is a partition of n, where $\boxplus$ denotes the Langlands sum (coming from his theory of *Eisenstein series*); moreover, every *constituent* π_j is unique up to isomorphism. Let $\mathcal{A}(n, \mathbb{Q})$ denote the set of isobaric automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ up to equivalence. Every isobaric π has an associated L-function $L(s, \pi) = \prod_v L(s, \pi_v)$, which admits a meromorphic continuation and a functional equation. Concretely, one associates at every prime p where π is unramified, a conjugacy class $A(\pi)$ in $\mathrm{GL}(n, \mathbb{C})$, or equivalently, an unordered n-tuple $(\alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{n,p})$ of complex numbers so that

$$L(s, \pi_p) = \prod_{j=1}^{n} (1 - \alpha_{j,p} p^{-s})^{-1}.$$

If π is cuspidal and non-trivial, $L(s, \pi)$ is entire; so is the incomplete one $L^S(s, \pi)$ for any finite set S of places of $\mathbb{Q}$.

Now suppose ρ_ℓ is an n-dimensional, semisimple ℓ-adic representation of $\mathcal{G}_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *corresponds* to an automorphic representation π of $\mathrm{GL}(n, \mathbb{A})$. We will take this to mean that there is a finite set S of places including ℓ, ∞ and all the primes where ρ_ℓ or π is ramified, such that we have

$$L(s, \pi_p) = L_p(s, \rho_\ell), \quad \forall p \notin S, \tag{0.1}$$

where the Galois Euler factor on the right is given by the characteristic polynomial of Fr_p, the Frobenius at p, acting on ρ_ℓ. When (0.1) holds (for a suitable S), we will write

$$\rho_\ell \leftrightarrow \pi.$$

A natural question in such a situation is to ask if π is cuspidal when ρ_ℓ is irreducible, and *vice versa*. It is certainly what is predicted by the general philosophy. However, proving it is another matter altogether, and positive evidence is scarce beyond $n = 2$.

One can answer this question in the affirmative, for any n, if one restricts to those ρ_ℓ which have *finite* image. In this case, it also defines a continuous, $\mathbb{C}$-representation ρ, the kind studied by E. Artin ([A]). Indeed, the hypothesis implies the identity of L-functions

$$L^S(s, \rho \otimes \rho^\vee) = L^S(s, \pi \times \pi^\vee), \tag{0.2}$$

where the superscript S signifies the removal of the Euler factors at places in S, and $\rho^\vee$ (resp. $\pi^\vee$) denotes the contragredient of ρ (resp. π). The L-function on the right is the Rankin–Selberg L-function, whose mirific properties have been established in the independent and complementary works of Jacquet, Piatetski-Shapiro and Shalika ([JPSS], and of Shahidi ([Sh1, Sh2]); see also [MW]. A theorem of Jacquet and Shalika ([JS1]) asserts that the *order of pole* at $s = 1$ of $L^S(s, \pi \times \pi^\vee)$ is 1 iff π is cuspidal. On the other hand, for any finite-dimensional $\mathbb{C}$-representation τ of $\mathcal{G}_\mathbb{Q}$, one has

$$-\mathrm{ord}_{s=1} L^S(s, \tau) = \dim_\mathbb{C} \mathrm{Hom}_{\mathcal{G}_\mathbb{Q}}(\underline{1}, \tau), \tag{0.3}$$

where $\underline{1}$ denotes the trivial representation of $\mathcal{G}_\mathbb{Q}$. Applying this with $\tau = \rho \otimes \rho^\vee \simeq \mathrm{End}(\rho)$, we see that the order of pole of $L^S(s, \rho \otimes \rho^\vee)$ at $s = 1$ is 1 iff the only operators in $\mathrm{End}(\rho)$ which commute with the $\mathcal{G}_\mathbb{Q}$-action are scalars, which means by Schur that ρ is irreducible. Thus, *in the Artin case, π is cuspidal iff ρ_ℓ is irreducible.*

For general ℓ-adic representations ρ_ℓ of $\mathcal{G}_\mathbb{Q}$, the order of pole at the right edge is not well understood. When ρ_ℓ comes from *arithmetic geometry*, i.e., when it is a Tate twist of a piece of the cohomology of a smooth projective variety over $\mathbb{Q}$ which is cut out by algebraic projectors, an important *conjecture of Tate* asserts an analogue of (0.3) for $\tau = \rho_\ell \otimes \rho_\ell^\vee$, but this is unknown except in a few families of examples, such as those coming from the theory of *modular curves*, *Hilbert modular surfaces* and *Picard modular surfaces*. So one has to find a different way to approach the problem, which works at least in low dimensions.

The main result of this paper is the following:

Theorem A. *Let* $n \leq 5$ *and let* ℓ *be a prime. Suppose* $\rho_\ell \leftrightarrow \pi$*, for an isobaric, algebraic automorphic representation* π *of* $GL(n, \mathbb{A})$*, and a continuous,* ℓ*-adic representation* ρ_ℓ *of* $\mathcal{G}_{\mathbb{Q}}$*. Assume*

(i) ρ_ℓ *is irreducible*
(ii) π *is odd if* $n \geq 3$
(iii) π *is semi-regular if* $n = 4$*, and regular if* $n = 5$

Then π *is cuspidal.*

Some words of explanation are called for at this point. An isobaric automorphic representation π is said to be *algebraic* ([Cℓ1]) if the restriction of $\sigma_\infty := \sigma(\pi_\infty(\frac{1-n}{2}))$ to $\mathbb{C}^*$ is of the form $\oplus_{j=1}^n \chi_j$, with each χ_j algebraic, i.e., of the form $z \to z^{p_j}\bar{z}^{q_j}$ with $p_j, q_j \in \mathbb{Z}$. (We do not assume that our automorphic representations are unitary, and the arrow $\pi_\infty \to \sigma_\infty$ is normalized arithmetically.) For $n = 1$, an algebraic π is an idele class character of type A_0 in the sense of Weil. One says that π is *regular* iff $\sigma_\infty|_{\mathbb{C}^*}$ is a direct sum of characters χ_j, each occurring with *multiplicity one*. And π is *semi-regular* ([BHR]) if each χ_j occurs with *multiplicity at most two*. Suppose ξ is a 1-dimensional representation of $W_{\mathbb{R}}$. Then, since $W_{\mathbb{R}}^{ab} \simeq \mathbb{R}^*$, ξ is defined by a character of $\mathbb{R}^*$ of the form $x \to |x|^w \cdot sgn(x)^{a(\xi)}$, with $a(\xi) \in \{0, 1\}$; here sgn denotes the sign character of $\mathbb{R}^*$. For every w, let $\sigma_\infty[\xi] := \sigma(\pi_\infty(\frac{1-n}{2}))[\xi]$ denote the isotypic component of ξ, which has dimension at most 2 (resp. 1) if π is semi-regular (resp. regular), and is acted on by $\mathbb{R}^*/\mathbb{R}^*_+ \simeq \{\pm 1\}$. We will call a semi-regular π *odd* if for every character ξ of $W_{\mathbb{R}}$, the eigenvalues of $\mathbb{R}^*/\mathbb{R}^*_+$ on the ξ-isotypic component are distinct. Clearly, any regular π is odd under this definition. See Section 1 for a definition of this concept for any algebraic π, not necessarily semi-regular.

I want to thank the organizers, Jae-Hyun-Yang in particular, and the staff, of the *International Symposium on Representation Theory and Automorphic Forms* in Seoul, Korea, first for inviting me to speak there (during February 14–17, 2005), and then for their hospitality while I was there. The talk I gave at the conference was on a different topic, however, and dealt with my ongoing work with Dipendra Prasad on *selfdual representations*. I would also like to thank F. Shahidi for helpful conversations and the referee for his comments on an earlier version, which led to an improvement of the presentation. It is perhaps apt to end this introduction at this point by acknowledging support from the National Science Foundation via the grant DMS – 0402044.

1 Preliminaries

1.1 Galois representations

For any field k with algebraic closure $\overline{k}$, denote by $\mathcal{G}_k$ the *absolute Galois group* of $\overline{k}$ over k. It is a projective limit of the automorphism groups of finite Galois extensions

E/k. We furnish $\mathcal{G}_k$ as usual with the *profinite topology*, which makes it a *compact, totally disconnected topological group*. When $k = \mathbb{F}_p$, there is for every n a unique extension of degree n, which is Galois, and $\mathcal{G}_{\mathbb{F}_p}$ is isomorphic to $\widehat{\mathbb{Z}} \simeq \lim_n \mathbb{Z}/n$, topologically generated by the *Frobenius automorphism* $x \to x^p$.

At each prime p, let $\mathcal{G}_p$ denote the local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with inertia subgroup I_p, which fits into the following exact sequence:

$$1 \to I_p \to \mathcal{G}_p \to \mathcal{G}_{\mathbb{F}_p} \to 1. \tag{1.1.1}$$

The fixed field of $\overline{\mathbb{Q}}_p$ under I_p is the *maximal unramified extension* $\mathbb{Q}_p^{\mathrm{ur}}$ of $\mathbb{Q}_p$, which is generated by all the roots of unity of order prime to p. One gets a natural isomorphism of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ with $\mathcal{G}_{\mathbb{F}_p}$. If $K/\mathbb{Q}$ is unramified at p, then one can lift the Frobenius element to a conjugacy class φ_p in $\mathrm{Gal}(K/\mathbb{Q})$.

All the Galois representations considered here will be continuous and finite-dimensional. Typically, we will fix a prime ℓ, and algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, and consider a continuous homomorphism

$$\rho_\ell : \mathcal{G}_{\mathbb{Q}} \to \mathrm{GL}(V_\ell), \tag{1.1.2}$$

where V_ℓ is an n-dimensional vector space over $\overline{\mathbb{Q}}_\ell$. We will be interested only in those ρ_ℓ that are unramified outside a finite set S of primes. Then ρ_ℓ factors through a representation of the quotient group $\mathcal{G}_S := G(\mathbb{Q}_S/\mathbb{Q})$, where $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ which is unramified outside S. One has the Frobenius classes ϕ_p in $\mathcal{G}_S$ for all $p \notin S$, and this allows one to define the L-factors (with $s \in \mathbb{C}$)

$$L_p(s, \rho_\ell) = \det(I - \varphi_p p^{-s} | V_\ell)^{-1}. \tag{1.1.3}$$

Clearly, it is the reciprocal of a polynomial in p^{-s} of degree n, with constant term 1, and it depends only on the equivalence class of ρ_ℓ. One sets

$$L^S(s, \rho_\ell) = \prod_{p \notin S} L_p(s, \rho_\ell). \tag{1.1.4}$$

When ρ_ℓ is the trivial representation, it is unramified everywhere, and $L^S(s, \rho_\ell)$ is none other than the *Riemann zeta function*. To define the *bad factors* at p in $S - \{\ell\}$, one replaces V_ℓ in (1.1.3) the subspace $V_\ell^{I_p}$ of *inertial invariants*, on which φ_p acts.

We are primarily interested in *semisimple representations* in this article, which are direct sums of *simple* (or *irreducible*) representations. Given any representation ρ_ℓ of $\mathcal{G}_{\mathbb{Q}}$, there is an associated *semisimplification*, denoted ρ_ℓ^{ss}, which is a direct sum of the simple Jordan–Holder components of ρ_ℓ. A *theorem of Tchebotarev* asserts the density of the Frobenius classes in the Galois group, and since the local p-factors of $L(s, \rho_\ell)$ are defined in terms of the *inverse roots* of φ_p, one gets the following standard, but useful result.

Proposition 1.1.5. *Let ρ_ℓ, ρ'_ℓ be continuous, n-dimensional ℓ-adic representations of $\mathcal{G}_{\mathbb{Q}}$. Then*

$$L^S(s, \rho_\ell) = L^S(s, \rho'_\ell) \implies \rho_\ell^{\mathrm{ss}} \simeq \rho_\ell'^{\mathrm{ss}}.$$

The Galois representations ρ_ℓ which have *finite image* are special, and one can view them as continuous $\mathbb{C}$-representations ρ. Artin studied these in depth and showed, using the results of Brauer and Hecke, that the corresponding L-functions admit meromorphic continuation and a functional equation of the form

$$L^*(s,\rho) = \varepsilon(s,\rho)L^*(1-s,\rho^\vee), \tag{1.1.6}$$

where $\rho^\vee$ denotes the contragredient representation on the dual vector space, where

$$L^*(s,\rho) = L(s,\rho)L_\infty(s,\rho), \tag{1.1.7}$$

with the *archimedean factor* $L_\infty(s,\rho)$ being a suitable product (shifted) gamma functions. Moreover,

$$\varepsilon(s,\rho) = W(\rho)N(\rho)^{s-1/2}, \tag{1.1.8}$$

which is an entire function of s, with the (non-zero) $W(\rho)$ being called the *root number* of ρ. The scalar $N(\rho)$ is an integer, called the *Artin conductor* of ρ, and the finite set S which intervenes is the set of primes dividing $N(\rho)$. The functional equation shows that $W(\rho)W(\rho^\vee) = 1$, and so $W(\rho) = \pm 1$ when ρ is *selfdual* (which means $\rho \simeq \rho^\vee$). Here is a useful fact:

Proposition 1.1.9 ([T]). *Let τ be a continuous, finite-dimensional $\mathbb{C}$-representation of $\mathcal{G}_{\mathbb{Q}}$, unramified outside S. Then we have*

$$-\mathrm{ord}_{s=1} L^S(s,\tau) = \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(\underline{1},\tau).$$

Corollary 1.1.10. *Let ρ be a continuous, finite-dimensional $\mathbb{C}$-representation of $\mathcal{G}_{\mathbb{Q}}$, unramified outside S. Then ρ is irreducible if and only if the incomplete L-function $L^S(s,\rho\otimes\rho^\vee)$ has a simple pole at $s = 1$.*

Indeed, if we set

$$\tau := \rho\otimes\rho^\vee \simeq \mathrm{End}(\rho), \tag{1.1.11}$$

then Proposition 1.1.9 says that the *order of pole* of $L(s,\rho\otimes\rho^\vee)$ at $s = 1$ is the *multiplicity of the trivial representation* in $\mathrm{End}(\rho)$ is 1, i.e., iff the *commutant* $\mathrm{End}_{\mathcal{G}_{\mathbb{Q}}}(\rho)$ is one-dimensional (over $\mathbb{C}$), which in turn is equivalent, by Schur's lemma, to ρ being irreducible. Hence the corollary.

For general ℓ-adic representations ρ_ℓ, there is no known analogue of Proposition 1.1.9, though it is predicted to hold (at the right edge of absolute convergence) by a *conjecture of Tate* when ρ_ℓ comes from *arithmetic geometry* (see [Ra4], Section 1, for example). Tate's conjecture is only known in certain special situations, such as for *CM abelian varieties*. For the L-functions in Tate's set-up, say of motivic weight $2m$, one does not even know that they make sense at the *Tate point* $s = m + 1$, let alone know its order of pole there. Things get even harder if ρ_ℓ does not arise from a geometric situation. One cannot work in too general a setting, and at a minimum, one needs to require ρ_ℓ to have some good properties, such as being unramified outside a finite set S of primes. Fontaine and Mazur conjecture ([FoM]) that ρ_ℓ is *geometric* if it has this property (of being unramified outside a finite S) and is in addition *potentially semistable*.

1.2 Automorphic representations

Let F be a number field with adele ring $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$, equipped with the adelic absolute value $|\cdot| = |\cdot|_{\mathbb{A}}$. For every algebraic group G over F, let $G(\mathbb{A}_F) = G(F_\infty) \times G(\mathbb{A}_{F,f})$ denote the restricted direct product $\prod'_v G(F_v)$, endowed with the usual locally compact topology. Then $G(F)$ embeds in $G(\mathbb{A}_F)$ as a discrete subgroup, and if Z_n denotes the center of $\mathrm{GL}(n)$, the homogeneous space $\mathrm{GL}(n,F)Z_n(\mathbb{A}_F)\backslash\mathrm{GL}(n,\mathbb{A}_F)$ has finite volume relative to the relatively invariant quotient measure induced by a Haar measure on $\mathrm{GL}(n,\mathbb{A}_F)$. An irreducible representation π of $\mathrm{GL}(n,\mathbb{A}_F)$ is admissible if it admits a factorization as a restricted tensor product $\otimes'_v \pi_v$, where each π_v is admissible and for almost all finite places v, π_v is *unramified*, i.e., has a no-zero vector fixed by $K_v = \mathrm{GL}(n,\mathcal{O}_v)$. (Here, as usual, $\mathcal{O}_v$ denotes the ring of integers of the local completion F_v of F at v.)

Fixing a unitary idele class character ω, which can be viewed as a character of $Z_n(\mathbb{A}_F)$, we may consider the space

$$L^2(n,\omega) := L^2(\mathrm{GL}(n,F)Z_n(\mathbb{A}_F)\backslash\mathrm{GL}(n,\mathbb{A}_F),\omega), \qquad (1.2.1)$$

which consists of (classes of) functions on $\mathrm{GL}(n,\mathbb{A}_F)$ that are left-invariant under $\mathrm{GL}(n,F)$, transform under $Z_n(\mathbb{A}_F)$ according to ω, and are square-integrable modulo $\mathrm{GL}(n,F)Z(\mathbb{A}_F)$. Clearly, $L^2(n,\omega)$ is a unitary representation of $\mathrm{GL}(n,\mathbb{A}_F)$ under the right translation action on functions. The **space of cusp forms**, denoted $L^2_0(n,\omega)$, consists of functions φ in $L^2(n,\omega)$ which satisfy the following for *every* unipotent radical U of a standard parabolic subgroup $P = MU$:

$$\int_{U(F)\backslash U(\mathbb{A}_F)} \varphi(ux) = 0. \qquad (1.2.2)$$

To say that P is a standard parabolic means that it contains the *Borel subgroup* of upper triangular matrices in $\mathrm{GL}(n)$. A basic fact asserts that $L^2_0(n,\omega)$ is a subspace of the discrete spectrum of $L^2(n,\omega)$.

By a **unitary cuspidal** (automorphic) representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, we will mean an irreducible, unitary representation occurring in $L^2_0(n,\omega)$. We will, by abuse of notation, also denote the underlying admissible representation by π. (To be precise, the unitary representation is on the Hilbert space completion of the admissible space.) Roughly speaking, unitary automorphic representations of $\mathrm{GL}(n,\mathbb{A}_F)$ are those which appear weakly in $L^2(n,\omega)$ for some ω. We will refrain from recalling the definition precisely, because we will work totally with the subclass of *isobaric automorphic representations*, for which one can take Theorem 1.2.10 (of Langlands) below as their definition.

If π is an admissible representation of $\mathrm{GL}(n,\mathbb{A}_F)$, then for any $z \in \mathbb{C}$, we define the *analytic Tate twist* of π by z to be

$$\pi(z) := \pi \otimes |\cdot|^z, \qquad (1.2.3)$$