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August 23–September 3, 2004
Novosibirsk, Russia

V. I. Burenkov
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The Interaction of Analysis and Geometry

Preface

The volume contains the papers of participants of the International Conference on Analysis and Geometry in honor of the 75th birthday of Yuriĭ Reshetnyak, an outstanding mathematician, a member of the Russian Academy of Science. His prominent investigations of two-dimensional manifolds of bounded curvature and mappings with bounded distortion are an outstanding contribution into the mathematics. The conference was held through August 23 – September 3, 2004 in Novosibirsk.

The scope of the volume includes geometry of spaces with bounded curvature in the sense of Alexandrov, quasiconformal mappings and mappings with bounded distortion (quasiregular mappings), nonlinear potential theory, Sobolev spaces, spaces with fractional and generalized smoothness, and variational problems.

The volume reflects modern trends in these areas. Most articles are related to Reshetnyak's original works and demonstrate the vitality of his fundamental contribution in some important fields of mathematics such as the geometry in the "large", quasiconformal analysis, Sobolev spaces, potential theory and variational calculus.

The topics discussed in the volume can be subject of research for many mathematicians. We hope that this volume will be a valuable source both for experts and young researchers by providing them with a wealth of information, which otherwise is scattered in literature or not published at all.

V. Burenkov, T. Iwaniec and S. Vodopyanov
September, 2006

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On an Extremal Property of Quadrilaterals in an Aleksandrov Space of Curvature $\leq K$

I. D. Berg and I. G. Nikolaev

Dedicated to Yuriĭ Reshetnyak on his 75th birthday

Abstract. In this note we introduce an analog of the notion of the cosine of the angle between two "directions", possibly based at different points of a metric space. For two pairs of points, we introduce the notion of the K-quadrilateral cosine, cosq_K; in a space of constant curvature, it coincides with the actual cosine of the angle between two tangent vectors under Levi-Civita parallel translation. We prove that in an $\Re_K$ domain of an Aleksandrov space of curvature $\leq K$, we have $|\operatorname{cosq}_K| \leq 1$. Our principal result states: if, for a quadrilateral with two non-adjacent "directed" sides of equal length in an $\Re_K$ domain, we have $\operatorname{cosq}_K = -1$ for those two sides, then the geodesic convex hull of the quadrilateral is isometric to the geodesic convex hull of a K-parallelogramoid in a two-dimensional space of constant curvature K.

1. Introduction

The distance between two tangent vectors to a Riemannian space, possibly based at different points, is given by the *Sasaki metric* [**S1, S2**]. A generalization of the Sasaki distance to general metric spaces was given in [**N1, N2**]. The starting point of our definition of the Sasaki metric in an abstract metric space is the *quadrilateral cosine.* We will keep the notation $\overrightarrow{u} = \overrightarrow{AB}$ for an ordered pair (A, B) in a metric space $(\mathcal{M}, \rho)$. If $\mathcal{Q} = \{A, B, C, D\}$ is a quadruple of points of $\mathcal{M}$, $A \neq B$, $C \neq D$, then we define the quadrilateral cosine by

$$\operatorname{cosq}(\overrightarrow{AB}, \overrightarrow{CD}) = \frac{\rho^2(A, D) + \rho^2(C, B) - \rho^2(A, C) - \rho^2(B, D)}{2\rho(A, B)\rho(C, D)},$$

which equals the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in Euclidean space.

In a general metric space, the quadrilateral cosine can be greater than one. In our paper [**BeN**], we have shown that the condition of $|\operatorname{cosq}|$ being not greater than one is closely related to the nonpositiveness of the curvature of the metric space in the sense of A. D. Aleksandrov. In particular, in an $\Re_0$ domain of an

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Aleksandrov space of nonpositive curvature, $|\text{cosq}| \leq 1$ and if, for some quadruple, $\text{cosq} = 1$, then the geodesic convex hull of that quadruple is either isometric to a Euclidean trapezoid or to a segment of straight line.

In this paper, we prove similar results for Aleksandrov spaces of curvature bounded from above by K. We introduce the *K-quadrilateral cosine.* We prove that in an $\Re_K$ domain of a space of curvature $\leq K$, the K-quadrilateral cosine does not exceed 1 in absolute value.

We call a quadruple $\mathcal{Q} = \{A, B, P, P'\}$ of distinct points in a metric space $(\mathcal{M}, \rho)$ a *K-parallelogramoid* if $\text{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BP'}\right) = -1$ and $\rho(A, P) = \rho(B, P')$. In a Riemannian space of constant curvature, our definition of the parallelogramoid is slightly different from that of the classical Levi-Civita parallelogramoid [**C**, Sec. 229].

The main theorem of this paper is an extremal theorem stating that the geodesic convex hull of a K-parallelogramoid in an $\Re_K$ domain of a space of curvature $\leq K$ is isometric to the geodesic convex hull of a K-parallelogramoid in a two-dimensional space of constant curvature K.

2. Aleksandrov space of curvature $\leq K$

A. D. Aleksandrov introduced spaces of curvature $\leq K$ in his papers [**A1, A2**].

A metric ρ of the metric space $(\mathcal{M}, \rho)$ is called *intrinsic*, if for every $P, Q \in \mathcal{M}$,

$$\rho(P, Q) = \inf_{\mathcal{L}}\{\ell_\rho(\mathcal{L})\},$$

where inf is taken over all curves $\mathcal{L}$ joining the points P and Q, and $\ell_\rho(\mathcal{L})$ is the length of $\mathcal{L}$ measured in the metric ρ.

A curve $\mathcal{L}$ in a metric space $(\mathcal{M}, \rho)$ joining a pair of points A, B is called a *shortest arc* if its length is equal to $\rho(A, B)$.

A metric space is said to be *geodesically connected* if each pair of points in it can be joined by a shortest arc.

A configuration consisting of three distinct points (*vertices*) and three shortest arcs joining these points pairwise (*sides*) is called a (*geodesic*) *triangle.* We will use a convenient notation of Euclidean geometry ABC or ΔABC to denote the triangle $\mathcal{T}$, and $\mathcal{AB}$, $\mathcal{BC}$ and $\mathcal{AC}$ to denote its sides and AB, BC, AC to denote the corresponding lengths. The *perimeter* $p(\mathcal{T})$ of a triangle $\mathcal{T} = ABC$ is the sum $AB + BC + AC$.

The *K-plane* $\mathbb{S}_K$ is the Euclidean plane if $K = 0$, the open hemisphere of radius $1/\sqrt{K}$ if $K > 0$ and the Lobatchevskii plane of curvature K, if $K < 0$. The definition of n-dimensional K-space $\mathbb{S}_K^n$ is similar.

If $\mathcal{T} = ABC$ is a triangle in a metric space, its *isometric copy* in the K-plane is the triangle $\mathcal{T}^K = A^K B^K C^K$ in $\mathbb{S}_K$ having the same side lengths as $\mathcal{T}$:

$$AB = A^K B^K, \quad AC = A^K C^K \quad \text{and} \quad BC = B^K C^K.$$

If $K > 0$ we require that the perimeter of $\mathcal{T}$ be less than $2\pi/\sqrt{K}$.

Let $\mathcal{L}$ and $\mathcal{N}$ be two shortest arcs with a common starting point O in a metric space $(\mathcal{M}, \rho)$. Let $X \in \mathcal{L}$ and $Y \in \mathcal{N}$, where $X \neq O$ and $Y \neq O$. Set $x = OX$ and $y = OY$. Let $\mathcal{T}^K = O^K X^K Y^K$ be the isometric copy of the triangle $\mathcal{T} = OXY$ in the K-plane. Then $\gamma_{\mathcal{L}\mathcal{N}}^K(x, y)$ denotes the angle of the triangle $\mathcal{T}^K$ at its vertex O^K.

The *upper angle* between the curves $\mathcal{L}$ and $\mathcal{N}$ is defined by

$$\overline{\alpha}(\mathcal{L},\mathcal{N}) = \varlimsup_{x\to 0,\, y\to 0} \gamma^0_{\mathcal{LN}}(x,y).$$

In a metric space $(\mathcal{M},\rho)$, the *excess* of a triangle $\mathcal{T} = ABC$ is defined by

$$\delta(\mathcal{T}) = \overline{\alpha} + \overline{\beta} + \overline{\gamma} - \pi.$$

The *excess* of $\mathcal{T}$ *with respect to* K is

$$\delta_K(\mathcal{T}) = (\overline{\alpha} + \overline{\beta} + \overline{\gamma}) - (\alpha_K + \beta_K + \gamma_K),$$

where α_K, β_K, γ_K are the corresponding angles of an isometric copy $\mathcal{T}^K$ of the triangle $\mathcal{T}$ in the K-plane.

An $\Re_K$ *domain* (also known as a $CAT\,(K)$ space) is a metric space with the following properties:

(i) $\Re_K$ is a geodesically connected metric space.
(ii) If $K > 0$, then the perimeter of each triangle in $\Re_K$ is less than $2\pi/\sqrt{K}$.
(iii) Each triangle in $\Re_K$ has nonpositive excess with respect to K:

$$\delta_K(\mathcal{T}) \le 0.$$

A metric space $(\mathcal{M},\rho)$ is a *space of curvature* $\le K$ in the sense of A. D. Aleksandrov, if each point of $\mathcal{M}$ is contained in some neighborhood that is an $\Re_K$ domain.

We recall the following fundamental properties of spaces of curvature $\le K$.

ANGLE COMPARISON THEOREM [**A2**]: *the upper angles* $\overline{\alpha},\overline{\beta},\overline{\gamma}$ *of an arbitrary triangle* $\mathcal{T}$ *in* $\Re_K$ *are not greater than the corresponding angles* α_K,β_K *and* γ_K *of the triangle* $\mathcal{T}^K$ *on* $\mathbb{S}_K$, *i.e.,*

$$\overline{\alpha} \le \alpha_K, \overline{\beta} \le \beta_K, \overline{\gamma} \le \gamma_K.$$

K-CONCAVITY [**A2**] (also known as $CAT(K)$-inequality): *let* X, Y *be points on the sides* $\mathcal{AB}$ *and* $\mathcal{AC}$ *of the triangle* $\mathcal{T} = ABC$ *in a domain* $\Re_K$ *and let* $X' \in \mathcal{A}^K\mathcal{B}^K$ *and* $Y' \in \mathcal{A}^K\mathcal{C}^K$ *be points on the sides of the corresponding isometric copy* $\mathcal{T}^K = A^KB^KC^K$ *in* $\mathbb{S}_K$ *such that* $A^KX' = AX$, *and* $A^KY' = AY$. *Then* $XY \le X'Y'$.

These results were established by A. D. Aleksandrov in the 1950's.

In 1968 Yu. G. Reshetnyak proved a far reaching generalization of K-concavity [**R**]. Let $\mathcal{L}$ be a closed curve in a metric space $(\mathcal{M},\rho)$ such that $\ell_\rho(\mathcal{L}) < 2\pi/\sqrt{K}$ if $K > 0$. Let $\mathcal{V}$ be a convex domain in $\mathbb{S}_K$ with the bounding curve $\mathcal{N}$. We say that $\mathcal{V}$ *majorizes* the curve $\mathcal{L}$ if there is a non-expanding mapping of the domain $\mathcal{V}$ into $\mathcal{M}$ that maps $\mathcal{N}$ onto $\mathcal{L}$ and preserves arc length. The domain $\mathcal{V}$ is called the *majorant* for $\mathcal{L}$.

RESHETNYAK'S MAJORIZATION THEOREM: *In an* $\Re_K$ *domain, for any rectifiable closed curve* $\mathcal{L}$ *whose length is less than* $2\pi/\sqrt{K}$ *when* $K > 0$, *there is a convex domain in* $\mathbb{S}_K$ *that majorizes* $\mathcal{L}$.

3. K-quadrilateral cosine

By definition, the 0-quadrilateral cosine, cosq_0 is just cosq. In this section, we introduce the notion of the *K-quadrilateral cosine,* cosq_K, for $K \neq 0$.

First we describe the K-quadrilateral cosine in K-space. Consider a quadruple $\mathcal{Q} = \{A, B, P, Q\}$ of points, $A \neq P$ and $B \neq Q$, in $\mathbb{S}^3_K$. Let

$$AP = x,\ BQ = y,\ AB = a,\ PQ = b,\ PB = d \ \text{ and } AQ = f.$$

Consider two shortest arcs $\mathcal{AP}$ and $\mathcal{BQ}$. Let

$$\xi = \exp_A^{-1}(P)/AP \quad \text{and} \quad \zeta = \exp_B^{-1}(Q)/BQ.$$

Then, by definition,

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \cos\angle(\zeta, \xi''),$$

where the unit tangent vector ξ'' at the point B is parallel in K-space to the unit tangent vector ξ along the shortest arc $\mathcal{BA}$.

The following lemma expresses $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ in terms of six distances x, y, a, b, d and f. Let $K \neq 0$ and $\kappa = \sqrt{|K|}$. Set

$$\widehat{\kappa} = \begin{cases} \kappa = \sqrt{K} & \text{if } K > 0, \\ i\kappa = i\sqrt{-K} & \text{if } K < 0. \end{cases}$$

Consider a triangle $\mathcal{T} = ABC$ in the K-plane. Set $a = BC, b = AC$, $c = AB$ and $\alpha = \measuredangle BAC$. Notice that because

$$\sin ix = i\sinh(x), \quad \cos(ix) = \cosh x,$$

the cosine formula for both spherical and Lobatchevskii planes

$$\begin{cases} \cos\kappa a = \cos\kappa c\cos\kappa b + \sin\kappa c\sin\kappa b\cos\alpha & \text{if } K > 0, \\ \cosh\kappa a = \cosh\kappa c\cosh\kappa b - \sinh\kappa c\sinh\kappa b\cos\alpha & \text{if } K < 0 \end{cases}$$

can be written as

$$\cos\widehat{\kappa}a = \cos\widehat{\kappa}c\cos\widehat{\kappa}b + \sin\widehat{\kappa}c\sin\widehat{\kappa}b\cos\alpha.$$

LEMMA 3.1.

$$\begin{aligned} \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) &= (\cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y\cos\widehat{\kappa}a + \cos\widehat{\kappa}b\cos\widehat{\kappa}a \\ &- \cos\widehat{\kappa}x\cos\widehat{\kappa}f - \cos\widehat{\kappa}d\cos\widehat{\kappa}y - \cos\widehat{\kappa}d\cos\widehat{\kappa}f)/(1+\cos\widehat{\kappa}a)\sin\widehat{\kappa}x\sin\widehat{\kappa}y. \end{aligned} \tag{3.1}$$

PROOF. We will use the following constructive interpretation of parallel translation in $\mathbb{S}^3_K$. Let O be the midpoint of the shortest arc $\mathcal{AB}$ and let P' be a point symmetric to the point P relative to O, that is, O is the midpoint of the shortest arc $\mathcal{PP}'$, as illustrated in Figure 1. It is readily seen that in K-space, the unit vector ξ, tangent to $\mathcal{AP}$, is parallel in $\mathbb{S}^3_K$ to the unit tangent vector $\xi'' = -\xi'$ along the shortest arc $\mathcal{AB}$.

First, we calculate $\cos\widehat{\kappa}OQ$. By the cosine formula for the triangle ABQ,

$$\cos\widehat{\kappa}f = \cos\widehat{\kappa}a\cos\widehat{\kappa}y + \sin\widehat{\kappa}a\sin\widehat{\kappa}y\cos\angle ABQ,$$

whence

$$\cos\angle ABQ = \frac{\cos\widehat{\kappa}f - \cos\widehat{\kappa}a\cos\widehat{\kappa}y}{\sin\widehat{\kappa}a\sin\widehat{\kappa}y}.$$

$$\begin{aligned} \cos\widehat{\kappa}OQ &= \cos\widehat{\kappa}\frac{a}{2}\cos\widehat{\kappa}y + \sin\widehat{\kappa}\frac{a}{2}\sin\widehat{\kappa}y\cos\angle ABQ \\ &= \cos\widehat{\kappa}\frac{a}{2}\cos\widehat{\kappa}y + \frac{\cos\widehat{\kappa}f - \cos\widehat{\kappa}a\cos\widehat{\kappa}y}{2\cos\widehat{\kappa}\frac{a}{2}}. \end{aligned} \tag{3.2}$$

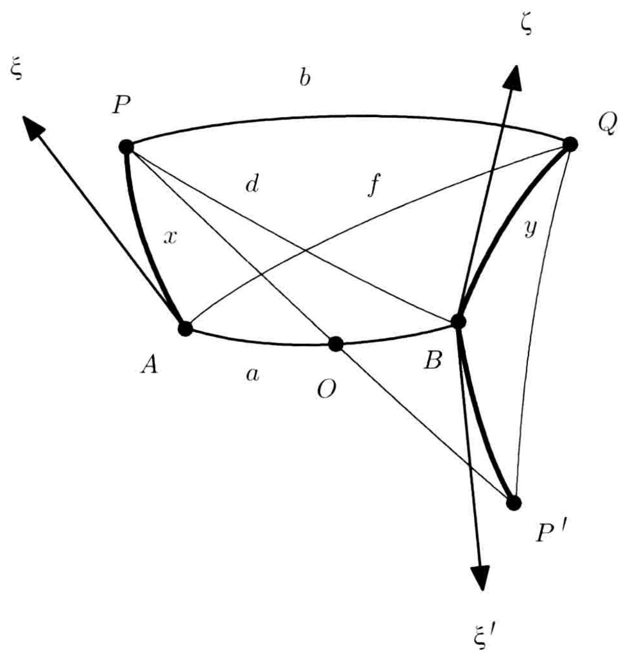

FIGURE 1. Computation of cosq_K in $\mathbb{S}^3_K$.

In a similar way, from triangles ABP and AOP, we find $\cos\widehat{\kappa}PO$:

$$\cos\widehat{\kappa}PO = \cos\widehat{\kappa}\frac{a}{2}\cos\widehat{\kappa}x + \frac{\cos\widehat{\kappa}d - \cos\widehat{\kappa}a\cos\widehat{\kappa}x}{2\cos\widehat{\kappa}\frac{a}{2}}. \tag{3.3}$$

Now, from the triangles OPQ and $P'PQ$, we find $\cos\widehat{\kappa}P'Q$:

$$\cos\angle OPQ = \frac{\cos\widehat{\kappa}OQ - \cos\widehat{\kappa}PO\cos\widehat{\kappa}b}{\sin\widehat{\kappa}PO\sin\widehat{\kappa}b} \quad\Longrightarrow$$

$$\begin{aligned}
\cos\widehat{\kappa}P'Q &= \cos 2\widehat{\kappa}PO\cos\widehat{\kappa}b + \sin 2\widehat{\kappa}PO\sin\widehat{\kappa}b\cos\angle OPQ\\
&= \cos 2\widehat{\kappa}PO\cos\widehat{\kappa}b + 2\cos\widehat{\kappa}PO\left(\cos\widehat{\kappa}OQ - \cos\widehat{\kappa}PO\cos\widehat{\kappa}b\right)\\
&= \left(2\cos^2\widehat{\kappa}PO - 1\right)\cos\widehat{\kappa}b + 2\cos\widehat{\kappa}PO\left(\cos\widehat{\kappa}OQ - \cos\widehat{\kappa}PO\cos\widehat{\kappa}b\right)\\
&= -\cos\widehat{\kappa}b + 2\cos\widehat{\kappa}PO\cos\widehat{\kappa}OQ.
\end{aligned}$$

From the triangle QBP', we get:

$$\cos\angle P'BQ = \frac{-\cos\widehat{\kappa}b + 2\cos\widehat{\kappa}PO\cos\widehat{\kappa}OQ - \cos\widehat{\kappa}x\cos\widehat{\kappa}y}{\sin\widehat{\kappa}x\sin\widehat{\kappa}y}.$$

Hence,

$$\begin{aligned}
\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) &= -\cos\angle P'BQ\\
&= \frac{\cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y - 2\cos\widehat{\kappa}PO\cos\widehat{\kappa}OQ}{\sin\widehat{\kappa}x\sin\widehat{\kappa}y}.
\end{aligned} \tag{3.4}$$

Consider separately the following term of the foregoing equation for cosq_K. By (3.2) and (3.3),

$$\cos\widehat{\kappa PO}\cos\widehat{\kappa OQ} = \left(\cos\widehat{\kappa x}\cos\widehat{\kappa}\frac{a}{2} + \frac{\cos\widehat{\kappa d} - \cos\widehat{\kappa x}\cos\widehat{\kappa a}}{2\cos\widehat{\kappa}\frac{a}{2}}\right)$$
$$\times\left(\cos\widehat{\kappa}\frac{a}{2}\cos\widehat{\kappa y} + \frac{\cos\widehat{\kappa f} - \cos\widehat{\kappa a}\cos\widehat{\kappa y}}{2\cos\widehat{\kappa}\frac{a}{2}}\right).$$

Simplification of the first factor yields:

$$\cos\widehat{\kappa x}\cos\widehat{\kappa}\frac{a}{2} + \frac{\cos\widehat{\kappa d} - \cos\widehat{\kappa x}\cos\widehat{\kappa a}}{2\cos\widehat{\kappa}\frac{a}{2}}$$
$$= \frac{(1+\cos\widehat{\kappa a})\cos\widehat{\kappa x} + \cos\widehat{\kappa d} - \cos\widehat{\kappa x}\cos\widehat{\kappa a}}{2\cos\widehat{\kappa}\frac{a}{2}} = \frac{\cos\widehat{\kappa x} + \cos\widehat{\kappa d}}{2\cos\widehat{\kappa}\frac{a}{2}}.$$

In a similar way,

$$\cos\widehat{\kappa}\frac{a}{2}\cos y + \frac{\cos\widehat{\kappa f} - \cos\widehat{\kappa a}\cos\widehat{\kappa y}}{2\cos\widehat{\kappa}\frac{a}{2}} = \frac{\cos\widehat{\kappa y} + \cos\widehat{\kappa f}}{2\cos\widehat{\kappa}\frac{a}{2}}.$$

So,

$$2\cos\widehat{\kappa PO}\cos\widehat{\kappa OQ}$$
$$= \frac{\cos\widehat{\kappa x}\cos\widehat{\kappa y} + \cos\widehat{\kappa x}\cos\widehat{\kappa f} + \cos\widehat{\kappa d}\cos\widehat{\kappa y} + \cos\widehat{\kappa d}\cos\widehat{\kappa f}}{(1+\cos\widehat{\kappa a})}. \tag{3.5}$$

By (3.4) and (3.5),

$$\cos\widehat{\kappa b} + \cos\widehat{\kappa x}\cos\widehat{\kappa y} - 2\cos\widehat{\kappa PO}\cos\widehat{\kappa OQ}$$
$$= \cos\widehat{\kappa b} + \cos\widehat{\kappa x}\cos\widehat{\kappa y}$$
$$- \frac{\cos\widehat{\kappa x}\cos\widehat{\kappa y} + \cos\widehat{\kappa x}\cos\widehat{\kappa f} + \cos\widehat{\kappa d}\cos\widehat{\kappa y} + \cos\widehat{\kappa d}\cos\widehat{\kappa f}}{1+\cos\widehat{\kappa a}}$$
$$= (\cos\widehat{\kappa b} + \cos\widehat{\kappa b}\cos\widehat{\kappa a} + \cos\widehat{\kappa x}\cos\widehat{\kappa y}\cos\widehat{\kappa a}$$
$$- \cos\widehat{\kappa x}\cos\widehat{\kappa f} - \cos\widehat{\kappa d}\cos\widehat{\kappa y} - \cos\widehat{\kappa d}\cos\widehat{\kappa f})\,/\,(1+\cos\widehat{\kappa a}).$$

Hence, the formula for $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ follows. □

DEFINITION 3.2. Let $(\mathcal{M}, \rho)$ be a metric space and $A, P, B, Q \in \mathcal{M}$ be such that $A \neq P, B \neq Q$. Let $\rho(A, B) < \pi/\sqrt{K}$, when $K > 0$. If

$$\rho(A,P) = x, \quad \rho(B,Q) = y, \quad \rho(A,B) = a,$$
$$\rho(P,Q) = b, \quad \rho(P,B) = d, \quad \rho(A,Q) = f \quad \text{and} \quad \kappa = \sqrt{|K|},$$

then the K-quadrilateral cosine $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ is defined by (3.1), that is,

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = [\cos\kappa b + \cos\kappa x\cos\kappa y\cos\kappa a + \cos\kappa b\cos\kappa a$$
$$- \cos\kappa x\cos\kappa f - \cos\kappa d\cos\kappa y - \cos\kappa d\cos\kappa f]/(1+\cos\kappa a)\sin\kappa x\sin\kappa y,$$

if $K > 0$, and

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = [\cosh\kappa x\cosh\kappa f + \cosh\kappa d\cosh\kappa y + \cosh\kappa d\cosh\kappa f$$
$$- \cosh\kappa b - \cosh\kappa x\cosh\kappa y\cosh\kappa a$$
$$- \cosh\kappa b\cosh\kappa a]/(1+\cosh\kappa a)\sinh\kappa x\sinh\kappa y,$$

if $K < 0$.

REMARK 3.3. It is easy to see that $\lim_{K\to 0} \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \operatorname{cosq}(\overrightarrow{AP}, \overrightarrow{BQ})$.

REMARK 3.4. If $A = B$, then $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{AQ}) = \cos\gamma^K_{\mathcal{LN}}(x, y)$, where $\mathcal{L} = \mathcal{AP}$, $\mathcal{N} = \mathcal{AQ}$ and $x = AP$, $y = AQ$.

In conclusion, we state the following convenient form of the definition of cosq_K.

LEMMA 3.5. *Let $K \neq 0$. Then $\mathbf{q} = \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ if and only if*

$$(1 + \cos\widehat{\kappa}a)(\cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y - \mathbf{q}\sin\widehat{\kappa}x\sin\widehat{\kappa}y) = (\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f).$$

In particular, $\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = -1$ if and only if

$$(1 + \cos\widehat{\kappa}a)\left[\cos\widehat{\kappa}b + \cos\widehat{\kappa}(x - y)\right] = (\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f).$$

PROOF. Notice that $\mathbf{q} = \operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ})$ is equivalent to the following equation:

$$\begin{aligned}\mathbf{q}(1 + \cos\widehat{\kappa}a)\sin\widehat{\kappa}x\sin\widehat{\kappa}y &= \cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y\cos\widehat{\kappa}a + \cos\widehat{\kappa}b\cos\widehat{\kappa}a\\ &\quad - \cos\widehat{\kappa}x\cos\widehat{\kappa}f - \cos\widehat{\kappa}d\cos\widehat{\kappa}y - \cos\widehat{\kappa}d\cos\widehat{\kappa}f.\end{aligned}$$

By adding to both sides of the foregoing equation

$$(1 + \cos\widehat{\kappa}a)(-\cos\widehat{\kappa}x\cos\widehat{\kappa}y - \cos\widehat{\kappa}b),$$

we get:

$$\begin{aligned}&(1 + \cos\widehat{\kappa}a)(\mathbf{q}\sin\widehat{\kappa}x\sin\widehat{\kappa}y - \cos\widehat{\kappa}x\cos\widehat{\kappa}y - \cos\widehat{\kappa}b)\\ &= (1 + \cos\widehat{\kappa}a)(-\cos\widehat{\kappa}x\cos\widehat{\kappa}y - \cos\widehat{\kappa}b) + \cos\widehat{\kappa}b + \cos\widehat{\kappa}x\cos\widehat{\kappa}y\cos\widehat{\kappa}a\\ &\quad + \cos\widehat{\kappa}b\cos\widehat{\kappa}a - \cos\widehat{\kappa}x\cos\widehat{\kappa}f - \cos\widehat{\kappa}d\cos\widehat{\kappa}y - \cos\widehat{\kappa}d\cos\widehat{\kappa}f\\ &= -(\cos\widehat{\kappa}x + \cos\widehat{\kappa}d)(\cos\widehat{\kappa}y + \cos\widehat{\kappa}f),\end{aligned}$$

and the claim of the lemma follows. □

4. K-quadrilateral cosine in an $\Re_K$ domain

In a metric space, $|\operatorname{cosq}_K|$ can be greater than 1.

EXAMPLE 4.1. On the set $\mathbb{R}^2$ we specify the norm $\|(x, y)\|_1 = |x| + |y|$. Let $\mu > 0$; if $K > 0$, we assume that $\mu < \pi/\sqrt{K}$. Let $A = (0, \mu)$, $P = (0, \mu + t)$, $B = (\mu, \mu)$ and $Q = (\mu, \mu + t)$. Then

$$\begin{aligned}&a = AB = \mu, \quad b = PQ = \mu, \quad d = BP = f = AQ = \mu + t,\\ &x = AP = y = BQ = t.\end{aligned}$$

By (3.1),

$$\operatorname{cosq}_K(\overrightarrow{AP}, \overrightarrow{BQ}) = \frac{\cos\widehat{\kappa}\mu + \cos^2\widehat{\kappa}t\cos\widehat{\kappa}\mu - 2\cos\widehat{\kappa}t\cos\widehat{\kappa}(\mu + t) + \cos^2\widehat{\kappa}\mu - \cos^2\widehat{\kappa}(\mu + t)}{(1 + \cos\widehat{\kappa}\mu)\sin^2\widehat{\kappa}t}.$$

Because

$$\cos\widehat{\kappa}\mu + \cos^2\widehat{\kappa}t\cos\widehat{\kappa}\mu - 2\cos\widehat{\kappa}t\cos\widehat{\kappa}(\mu + t) + \cos^2\widehat{\kappa}\mu - \cos^2\widehat{\kappa}(\mu + t) = (\widehat{\kappa}\sin 2\widehat{\kappa}\mu + 2\widehat{\kappa}\sin\widehat{\kappa}\mu)\,t + O(t^2)$$

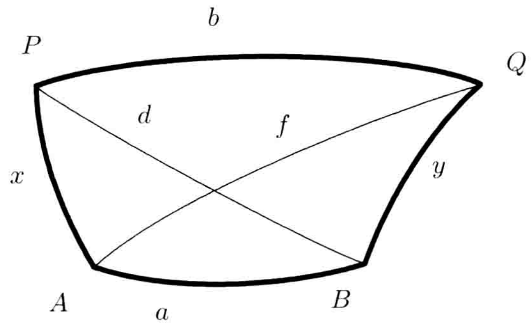

FIGURE 2. $\mathcal{L} = \mathcal{APQBA}$.

and

$$(1 + \cos\widehat{\kappa}\mu)\sin^2\widehat{\kappa}t = (1 + \cos\widehat{\kappa}\mu)\,\widehat{\kappa}^2 t^2 + O(t^4),$$

we see that $\left|\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right)\right|$ can be arbitrarily large, for small t.

We prove the following

THEOREM 4.2. *Let $\mathcal{Q} = \{A, B, P, Q\}$ be a quadruple of distinct points in an $\Re_K$ domain, $K \neq 0$. If diam $(\mathcal{Q}) < \pi/2\sqrt{K}$, when $K > 0$, then* $\left|\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right)\right| \leq 1$.

PROOF. Consider the closed curve $\mathcal{L} = \mathcal{APQBA}$ in the $\Re_K$ domain, as shown in Figure 2. If $K > 0$, then the length of $\mathcal{L}$ is less than $2\pi/\sqrt{K}$. By Reshetnyak's majorization theorem, there is a convex domain $\mathcal{V} \subset \mathbb{S}_K$ that majorizes $\mathcal{L}$. Then $\partial\mathcal{V}$ is a polygonal line $\mathcal{L}' = \mathcal{A'P'Q'B'A'}$ made of shortest arcs $\mathcal{A'P'}$, $\mathcal{P'Q'}$, $\mathcal{Q'B'}$ and $\mathcal{B'A'}$ in $\mathbb{S}_K$ such that $a = A'B'$, $x = A'P'$, $b = P'Q'$ and $y = B'Q'$. Because $\mathcal{V}$ majorizes $\mathcal{L}$, we also have:

$$d \leq d' = P'B' \quad \text{and} \quad f \leq f' = A'Q'.$$

We readily see from the definition of cosq_K that

$$\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right) \leq \operatorname{cosq}_K\left(\overrightarrow{A'P'}, \overrightarrow{B'Q'}\right).$$

Because $\mathcal{V} \subset \mathbb{S}_K$, by Lemma 3.1, we have $\left|\operatorname{cosq}_K\left(\overrightarrow{A'P'}, \overrightarrow{B'Q'}\right)\right| \leq 1$. So, the inequality

$$\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right) \leq 1$$

follows.

Now consider the closed curve $\mathcal{N} = \mathcal{AQBPA}$ in the $\Re_K$ domain, as shown in Figure 3. Notice that the length of $\mathcal{L}$ is less than $2\pi/\sqrt{K}$ when $K > 0$. Let a convex domain $\mathcal{U} \subset \mathbb{S}_K$ be a majorant for $\mathcal{N}$ with the polygonal boundary $\mathcal{A'Q'B'P'A'}$ satisfying $d = B'P' = d'$, $f = A'Q' = f'$, $x = A'P' = x'$ and $y = B'Q' = y'$. Because $\mathcal{U}$ majorizes $\mathcal{N}$, we have:

$$a \leq a' = A'B' \quad \text{and} \quad b \leq b' = P'Q'.$$

We claim that

$$\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right) \geq \operatorname{cosq}_K\left(\overrightarrow{A'P'}, \overrightarrow{B'Q'}\right).$$

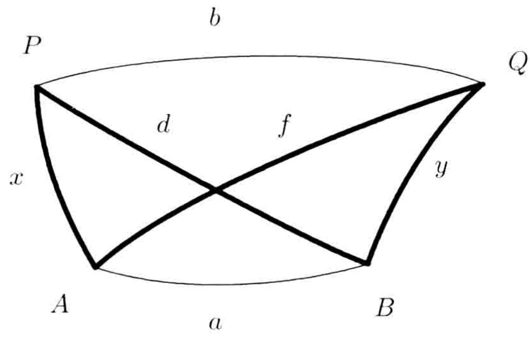

FIGURE 3. $\mathcal{N} = \mathcal{AQBPA}$.

Indeed, let $\mathbf{q} = \operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right)$ and $\mathbf{q}' = \operatorname{cosq}_K\left(\overrightarrow{A'P'}, \overrightarrow{B'Q'}\right)$. For definiteness, set $K > 0$. Then, by Lemma 3.5,

$$\begin{aligned} &(1+\cos\kappa a)(\cos\kappa b + \cos\kappa x\cos\kappa y - \mathbf{q}\sin\kappa x\sin\kappa y) \\ &= (\cos\kappa x + \cos\kappa d)\,(\cos\kappa y + \cos\kappa f) \\ &= (1+\cos\kappa a')\,(\cos\kappa b' + \cos\kappa x\cos\kappa y - \mathbf{q}'\sin\kappa x\sin\kappa y). \end{aligned}$$

Because $\cos\kappa a' \le \cos\kappa a$, we see that $(1+\cos\kappa a)\,/\,(1+\cos\kappa a') \ge 1$. Hence, by recalling the inequality $b \le b'$, we have:

$$\begin{aligned} &\frac{\cos\kappa b + \cos\kappa x\cos\kappa y - \mathbf{q}'\sin\kappa x\sin\kappa y}{\cos\kappa b + \cos\kappa x\cos\kappa y - \mathbf{q}\sin\kappa x\sin\kappa y} \\ &\ge \frac{\cos\kappa b' + \cos\kappa x\cos\kappa y - \mathbf{q}'\sin\kappa x\sin\kappa y}{\cos\kappa b + \cos\kappa x\cos\kappa y - \mathbf{q}\sin\kappa x\sin\kappa y} \ge 1, \end{aligned}$$

whence $\mathbf{q}' \le \mathbf{q}$ follows. The case of negative K is treated in a similar way.

To complete the proof of the theorem we observe that

$$\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right) \ge \operatorname{cosq}_K\left(\overrightarrow{A'P'}, \overrightarrow{B'Q'}\right) \ge -1. \qquad \square$$

5. Lemma concerning the parallelogramoid

LEMMA 5.1. *Let $K \ne 0$ and let $(\mathcal{M}, \rho)$ be a metric space with diameter less than $\pi/\sqrt{K}$ if $K > 0$ and such that $\left|\operatorname{cosq}_K\left(\overrightarrow{PQ}, \overrightarrow{RS}\right)\right| \le 1$, for every quadruple of distinct points $\{P, Q, R, S\}$ in $\mathcal{M}$. Let $\mathcal{Q} = \{A, B, P, P'\}$ be a K-parallelogramoid in $\mathcal{M}$; that is, $\mathcal{Q}$ is a quadruple of distinct points in $\mathcal{M}$ such that $AP = BP'$ and $\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BP'}\right) = -1$. Then*

$$\operatorname{cosq}_K\left(\overrightarrow{PA}, \overrightarrow{P'B}\right) = -1, \quad \operatorname{cosq}_K\left(\overrightarrow{PB}, \overrightarrow{P'A}\right) = -1, \quad \operatorname{cosq}_K\left(\overrightarrow{BP}, \overrightarrow{AP'}\right) = -1$$
$$\text{and} \quad PB = AP'$$

(see Figure 4).

PROOF. Let

$$x = AP = BP', \quad a = AB, \quad b = PP', \quad f = AP' \quad \text{and} \quad d = BP$$

as shown in the sketch. By Lemma 3.5, $\operatorname{cosq}_K\left(\overrightarrow{AP}, \overrightarrow{BP'}\right) = -1$ is equivalent to

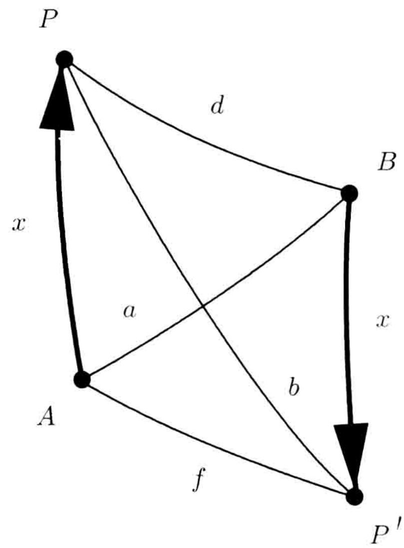

FIGURE 4. Parallelogramoid.

$$(\cos\widehat{\kappa}a+1)\,(\cos\widehat{\kappa}b+1)=(\cos\widehat{\kappa}x+\cos\widehat{\kappa}f)\,(\cos\widehat{\kappa}x+\cos\widehat{\kappa}d).$$

Hence, by Lemma 3.5, $\operatorname{cosq}_K(\overrightarrow{PA},\overrightarrow{P'B})=-1$, also.

Now we turn to the proof that $\operatorname{cosq}_K(\overrightarrow{PB},\overrightarrow{P'A})=-1$ and $PB=AP'$. By definition,

$$\begin{aligned}\operatorname{cosq}_K(\overrightarrow{PB},\overrightarrow{P'A}) &= \big\{\big[\cos\widehat{\kappa}a+\cos\widehat{\kappa}b\cos\widehat{\kappa}a-\cos\widehat{\kappa}f\cos\widehat{\kappa}x \\ &\quad -\cos\widehat{\kappa}d\cos\widehat{\kappa}x\big]+\cos\widehat{\kappa}b\cos\widehat{\kappa}f\cos\widehat{\kappa}d-\cos^2\widehat{\kappa}x\big\}\big/(1+\cos\widehat{\kappa}b)\sin\widehat{\kappa}f\sin\widehat{\kappa}d.\end{aligned}$$

Because $\operatorname{cosq}_K(\overrightarrow{AP},\overrightarrow{BP'})=-1$, we have

$$\cos\widehat{\kappa}a+\cos\widehat{\kappa}b+\cos\widehat{\kappa}a\cos\widehat{\kappa}b-\cos\widehat{\kappa}x\cos\widehat{\kappa}f-\cos\widehat{\kappa}x\cos\widehat{\kappa}d-\cos\widehat{\kappa}f\cos\widehat{\kappa}d = -\sin^2\widehat{\kappa}x.$$

So, $\cos\widehat{\kappa}a+\cos\widehat{\kappa}b\cos\widehat{\kappa}a-\cos\widehat{\kappa}f\cos\widehat{\kappa}x-\cos\widehat{\kappa}d\cos\widehat{\kappa}x=-\sin^2\widehat{\kappa}x+\cos\widehat{\kappa}f\cos\widehat{\kappa}d-\cos\widehat{\kappa}b$. Hence,

$$\begin{aligned}&\operatorname{cosq}_K(\overrightarrow{PB},\overrightarrow{P'A})\\ &=\frac{-\sin^2\widehat{\kappa}x+\cos\widehat{\kappa}f\cos\widehat{\kappa}d-\cos\widehat{\kappa}b+\cos\widehat{\kappa}b\cos\widehat{\kappa}f\cos\widehat{\kappa}d-\cos^2\widehat{\kappa}x}{(1+\cos\widehat{\kappa}b)\sin\widehat{\kappa}f\sin\widehat{\kappa}d}\\ &=\frac{-1+\cos\widehat{\kappa}f\cos\widehat{\kappa}d}{\sin\widehat{\kappa}f\sin\widehat{\kappa}d}.\end{aligned}$$

It is not difficult to see that

$$\operatorname{cosq}_K(\overrightarrow{PB},\overrightarrow{P'A})\le -1 \quad\text{and}\quad \operatorname{cosq}_K(\overrightarrow{PB},\overrightarrow{P'A})=-1 \iff f=d.$$