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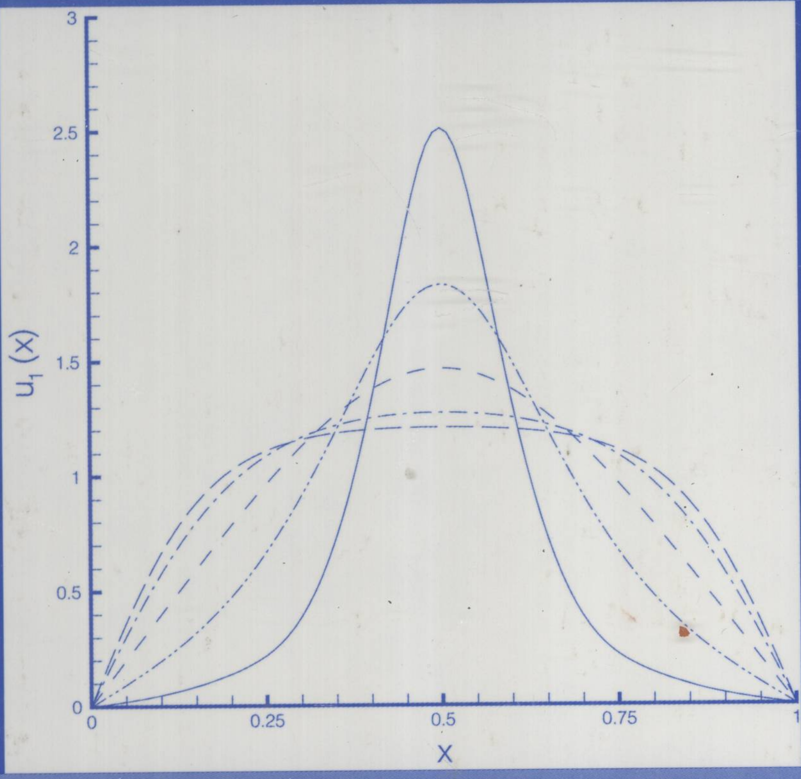
Beyond Perturbation

Introduction to the Homotopy Analysis Method



Shijun Liao

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To my wife, Shi Liu

Preface

In general it is difficult to obtain analytic approximations of nonlinear problems with strong nonlinearity. Traditionally, solution expressions of a nonlinear problem are mainly determined by the type of nonlinear equations and the employed analytic techniques, and the convergence regions of solution series are strongly dependent of physical parameters. It is well known that analytic approximations of nonlinear problems often break down as nonlinearity becomes strong and perturbation approximations are valid only for nonlinear problems with weak nonlinearity.

In this book we introduce an analytic method for nonlinear problems in general, namely the homotopy analysis method. We show that, even if a nonlinear problem has a unique solution, there may exist an *infinite* number of different solution expressions whose convergence region and rate are dependent on an auxiliary parameter. Unlike all previous analytic techniques, the homotopy analysis method provides us with a simple way to *control* and *adjust* the convergence region and rate of solution series of nonlinear problems. Thus, this method is valid for nonlinear problems with strong nonlinearity. Moreover, unlike all previous analytic techniques, the homotopy analysis method provides great freedom to use different base functions to express solutions of a nonlinear problem so that one can approximate a nonlinear problem more efficiently by means of better base functions. Furthermore, the homotopy analysis method logically contains some previous techniques such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method. Thus, it can be regarded as a unified or generalized theory of these previous methods.

The book consists of two parts. Part I (Chapter 1 to Chapter 5) deals with the basic ideas of the homotopy analysis method. In Chapter 2, the homotopy analysis method is introduced by means of a rather simple nonlinear problem. The reader is strongly advised to read this chapter first. In Chapter 3, a systematic description is given and a convergence theorem is described for general cases. In Chapter 4 we show that Lyapunov's artificial small parameter method, the δ-expansion method, and Adomian's decomposition method are simply special cases of the homotopy analysis method. In Chapter 5 the advantages and limitations of the homotopy analysis method are briefly discussed and some open questions are pointed out. In Part II (Chapter 6 to Chapter 18), the homotopy analysis method is applied to solve some nonlinear problems, such as simple bifurcations of a nonlinear boundary-value problem (Chapter 6), multiple solutions of a nonlinear boundary-value problem (Chapter 7), eigenvalue and eigenfunction of a nonlinear boundary-value problem (Chapter 8), the Thomas-Fermi atom model (Chapter 9), Volterra's population model (Chapter 10), free oscillations of conservative systems with odd nonlinearity (Chapter 11), free oscillations of conservative systems with quadratic nonlinearity (Chapter 12), limit cycle in a multidimensional system (Chapter 13), Blasius' viscous flow (Chapter 14), boundary-layer flows with exponential property (Chapter 15), boundary-layer flows with algebraic property (Chapter 16), Von Kármán swirling viscous flow (Chapter 17), and nonlinear progressive waves in deep water (Chapter 18). In Part II, only Chapters 14, 15, and 18 are adapted from published articles of the author.

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Finally, I would like to express my pure-hearted thanks to my wife for her love, understanding, and encouragement.

Contents

PART I

BASIC IDEAS

The way that can be spoken of is not the constant way;
The name that can be named is not the constant name.

Lao Tzu, an ancient Chinese philosopher

1

Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. Since the appearance of high-performance digit computers, it becomes easier and easier to solve a linear problem. However, generally speaking, it is still difficult to obtain accurate solutions of nonlinear problems. In particular, it is often more difficult to get an analytic approximation than a numerical one of a given nonlinear problem, although we now have high-performance supercomputers and some high-quality symbolic computation software such as Mathematica, Maple, and so on. The numerical techniques generally can be applied to nonlinear problems in complicated computation domain; this is an obvious advantage of numerical methods over analytic ones that often handle nonlinear problems in simple domains. However, numerical methods give discontinuous points of a curve and thus it is often costly and time consuming to get a complete curve of results. Besides, from numerical results, it is hard to have a whole and essential understanding of a nonlinear problem. Numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. The numerical and analytic methods of nonlinear problems have their own advantages and limitations, and thus it is unnecessary for us to do one thing and neglect another. Generally, one delights in giving analytic solutions of a nonlinear problem.

There are some analytic techniques for nonlinear problems, such as perturbation techniques [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] that are well known and widely applied. By means of perturbation techniques, a lot of important properties and interesting phenomena of nonlinear problems have been revealed. One of the astonishing successes of perturbation techniques is the discovery of the ninth planet in the solar system, found in the vast sky at a predicted point. Recently, the singular perturbation techniques are considered to be one of the top 10 progresses of theoretical and applied mechanics in the 20th century [13]. It is therefore out of question that perturbation techniques play important roles in the development of science and engineering. For further details, the reader is referred to the foregoing textbooks of perturbation methods.

Perturbation techniques are essentially based on the existence of small or large parameters or variables called perturbation quantity. Briefly speaking, perturbation techniques use perturbation quantities to transfer a nonlinear problem into an infinite number of linear sub-problems and then approximate it by the sum of solutions of the first several sub-problems. The existence of perturbation quantities is obviously a cornerstone of perturbation techniques, however, it is the perturbation quantity that brings perturbation techniques some serious restrictions. Firstly, it is impossible that every nonlinear problem contains such a perturbation quantity. This is an obvious restriction of perturbation techniques. Secondly, analytic approximations of nonlinear problems often break down as nonlinearity becomes strong, and thus perturbation approximations are valid only for nonlinear problems with weak nonlinearity. Consider the drag of a sphere in a uniform stream, a classical nonlinear problem in fluid mechanics governed by the famous Navier-Stokes equation, for example. Since 1851 when Stokes [14] first considered this problem, many scientists have attacked it by means of linear theories [15, 16], straightforward perturbation technique [17], and matching perturbation method [18, 19]. However, all these previous theoretical drag formulae agree with experimental data only for small Reynolds number, as shown in Figure 1.1. Thus, as pointed out by White [20], "the idea of using creeping flow to expand into the high Reynolds number region has not been successful". This might be partly due to the fact that perturbation techniques do not provide us with any ways to adjust convergence region and rate of perturbation approximations.

There are a few nonperturbation techniques. The dependence of perturbation techniques on small/large parameters can be avoided by introducing a so-called artificial small parameter. In 1892 Lyapunov [21] considered the equation

$$\frac{dx}{dt} = A(t)\, x,$$

where $A(t)$ is a time periodic matrix. Lyapunov [21] introduced an artificial parameter ϵ to replace this equation with the equation

$$\frac{dx}{dt} = \epsilon\, A(t)\, x$$

and then calculated power series expansions over ϵ for the solutions. In many cases Lyapunov proved that series converge for $\epsilon = 1$, and therefore we can put in the final expression by setting $\epsilon = 1$. The above approach is called Lyapunov's artificial small parameter method [21]. This idea was further employed by Karmishin et al. [22] to propose the so-called δ-expansion method. Karmishin et al. [22] introduced an artificial parameter δ to replace the equation

$$x^5 + x = 1 \tag{1.1}$$

with the equation

$$x^{1+\delta} + x = 1 \tag{1.2}$$

and then calculated power series expansions over δ and finally gained the approximations by converting the series to [3,3] Padé approximants and setting $\delta = 4$. In essence, the δ-expansion method is equivalent to the Lyapunov's artificial small parameter method. Note that both methods introduce an artificial parameter, although it appears in a different place and is denoted by different symbol in a given nonlinear equation. We additionally have great freedom to replace Equation (1.1) by many different equations such as

$$\delta\, x^5 + x = 1. \tag{1.3}$$

As pointed out by Karmishin et al. [22], the approximation given by the above equation is much worse than that given by Equation (1.2). Both the artificial small parameter method and the δ-expansion method obviously need some fundamental rules to determine the place where the artificial parameter ϵ or δ should appear. Like perturbation techniques, both the artificial small parameter method and the δ-expansion method themselves do not provide us with a convenient way to adjust convergence region and rate of approximation series.

Adomian's decomposition method [23, 24, 25] is a powerful analytic technique for strongly nonlinear problems. The basic ideas of Adomian's decomposition method is simply described in §4.1. Adomian's decomposition method is valid for ordinary and partial differential equations, no matter whether they contain small/large parameters, and thus is rather general. Moreover, the Adomian approximation series converge quickly. However, Adomian's decomposition method has some restrictions. Approximates solutions given by Adomian's decomposition method often contain polynomials. In general, convergence regions of power series are small, thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that power series is often not an efficient set of base functions to approximate a nonlinear problem, but unfortunately Adomain's decomposition method does not provide us with freedom to use different base functions. Like the artificial small parameter method and the δ-expansion method, Adomian's decomposition method itself also does not provide us with a convenient way to adjust convergence region and rate of approximation solutions.

In summary, neither perturbation techniques nor nonperturbation methods such as the artificial small parameter methods, the δ-expansion method, and Adomian's decomposition method can provide us with a convenient way to *adjust* and *control* convergence region and rate of approximation series. The efficiency to approximate a nonlinear problem has not been taken into enough account, therefore it is necessary to develop some new analytic methods such that they

1. Are valid for strongly nonlinear problems even if a given nonlinear problem does not contain any small/large parameters

2. Provide us with a convenient way to adjust the convergence region and rate of approximation series

3. Provide us with freedom to use different base functions to approximate a nonlinear problem.