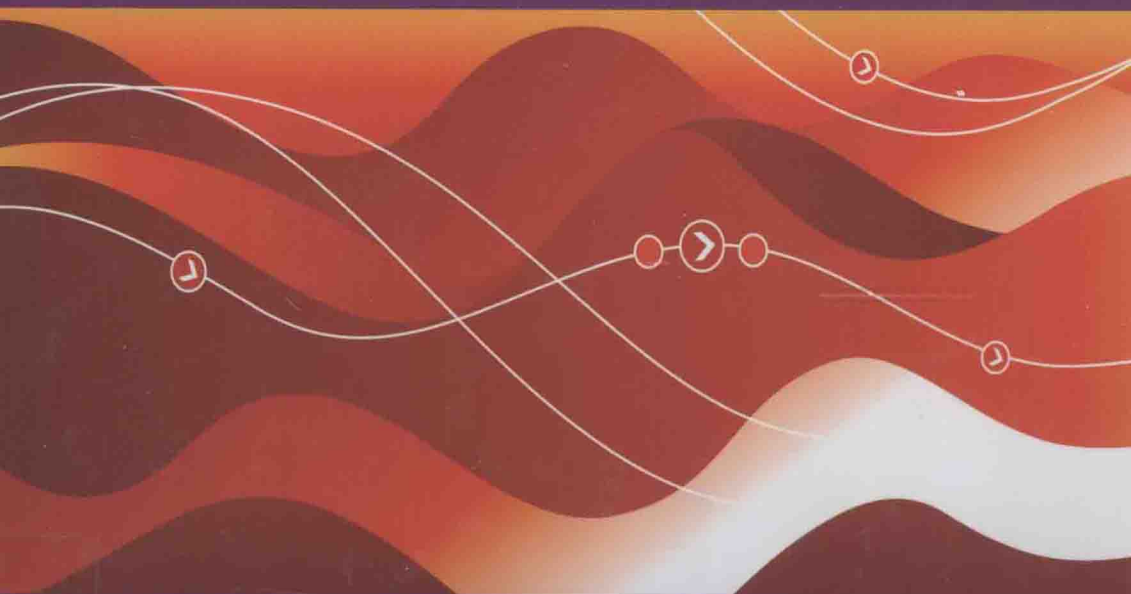


# Numerical Methods

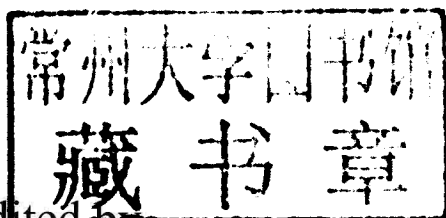
**Edited by Jean-Michel Tanguy**



Environmental Hydraulics  
*volume 3*

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# Numerical Methods



Edited by

Jean-Michel Tanguy

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## Numerical Methods

# Introduction

This environmental hydraulics treatise is made up of five volumes: Volume 1 describes the main physical processes and the physical domains where they can be observed and measured.

Volume 2 is dedicated to mathematical modeling in hydraulics and fluvial hydraulics.

In Volume 3, Chapters 1 to 7 constitute an introduction to numerical modeling, and more particularly on finite difference and finite element discretization. It in no way claims to constitute a treatise on the subject, but simply offers an overview of the discretization methods used in the domains covered by this work, which range from meteorology to shore morphodynamics. Chapters 8 to 13 deal with the finite volume discretization method, the spectral approach, numerical schemes and resolution methods.

Lastly, Volume 4 dealing with application examples completes Volume 3, along with a final volume (Volume 5) on operational software.

This volume is made up of three parts and comprises 13 chapters:

Part 1: general considerations concerning numerical tools;

Part 2: discretization methods;

Part 3: introduction to data assimilation.

Set out below is a brief summary of each chapter.

## Part 1: General considerations concerning numerical tools

We will introduce a number of general concepts regarding models used in engineering and in the operational-forecast domain and detail the ways of constructing numerical models based on mathematical models.

### Chapter outline

| Part title  | Chapter no. | Chapter title  | Problematic issue  |
|---|-------------|--|--|
| General Considerations Concerning Numerical Tools | 1           | Feedback on the Notion of a Model and the Need for Calibration | Placing perspective on the notion of a numerical model in the context of the study of physical phenomena. Importance of calibration  |
|   | 2           | Engineering Model and Real-Time Model                          | Transposing a model used in engineering into an operational forecast context requires significant computer-science and pairing work to be performed  |
|   | 3           | From Mathematical Model to Numerical Model                     | Switching from a mathematical model to a numerical model requires approximations to be performed; discretization methods suitable for the types of equations considered and suitable numerical schemes need to be used |

### What are the domain's perspectives?

– Operational forecast services such as the national meteorological services and flood forecasting services use real-time simulation tools based on numerical tools. *These tools need to be reliable, must not diverge and must be constantly recalibrated with respect to the reality in the field*, allowing civil security services and the general public to be warned of the imminence of a significant unforeseen event. Indeed, society's requirements are evolving towards a strong demand to be given preventative information as to risks, for there to be *greater risk-anticipation* and to be kept out of danger: we can cite as an example the mandatory preventive evacuation of the population of New Orleans when hurricane Gustav arrived in early September 2008, following the catastrophic events of Katrina in late August 2005.

– Significant progress has been made in recent years with respect to numerical modeling, underpinned by developments in computer science. This has enabled *complex geometries for very fine-scale studies* to be taken into account. Choosing

appropriate discretization methods and efficient schemes is a major challenge in engineering today. The decision makers of today are demanding increasingly higher standards with regard to technical choices and the use of tried and tested simulation tools, and only the most effective tools will last.

## Part two: Discretization methods

We will present the different numerical methods used within the domains covered by this book. Unlike a number of works dealing with these problems, we have opted not to remain focused on conceptual considerations, but to offer the reader a means of understanding the fundamentals of each method and their implementation. In particular, we explain the processing of boundary conditions, which are often overlooked. This lends something of a computational aspect to our presentations, but our aim is to provide the readers with the key principles, enabling them to follow the developments step by step.

### *Chapter outline*

| Part title             | No. | Section title                             | Problematic issue  |
|------------------------|-----|---|--|
| Discretization Methods | 4   | Problematic Issues Encountered            | Highlighting of several difficulties relative to the behavior of computing codes to demonstrate the importance of having efficient numerical schemes |
|                        | 5   | General Presentation of Numerical Methods | Placing perspective on the main existing numerical methods   |
|                        | 6   | Finite Differences                        | Succinct presentation of the method, illustrated using the equation for the diffusion of a pollutant   |
|                        | 7   | Introduction to the Finite Element Method | Detailed presentation of the method, illustrated using the equation for swell propagation  |
|                        | 8   | Presentation of the Finite Volume Method  | Detailed presentation of the method, illustrated using the equation for the development of a water table   |
|                        | 9   | Spectral methods in Meteorology           | This method is widely used in meteorological computing codes   |
|                        | 10  | Numerical-Scheme Study                    | Each discretization method requires a numerical scheme to be chosen, which must be studied to specify the behavior of the final model.               |
|                        | 11  | Resolution Methods                        | A brief list of the resolution methods   |

### ***What are the domain's perspectives?***

– The recent developments of numerical methods are mainly led by industrial applications. All of these methods, each with different origins, ultimately translate into the resolution of matrix systems. There are numerous links between them, and current research appears to be oriented towards methods, such as discontinuous finite element methods, which present a combination of the advantages of each of them.

– As we have mentioned on a number of occasions in the course of this book, the numerical tools of tomorrow will need to be equipped with high-level processing functionalities to offer the user the possibility of performing a reverse action at any instant on the resolution cycle.

### **Part three: Introduction to data assimilation**

This part presents the data assimilations methods that are most commonly used by forecast services.

The concepts on which these methods are based can appear somewhat abstruse, all the more so as the mathematical formulation is far from simple, but they represent powerful tools that are indispensable to forecasters to enable their models to adjust to the reality in the field.

We can expect these tools to undergo significant development in the coming years.

### ***Chapter outline***

| <b>Part title</b>                 | <b>Chapter no.</b> | <b>Chapter title</b>          | <b>Problematic issue</b>  |
|-----------------------------------|--------------------|-------------------------------|---|
| Introduction to Data Assimilation | 12                 | Data Assimilation             | General presentation of the various applications of the method: meteorology, hydrology and hydraulics |
|                                   | 13                 | Data Assimilation Methodology | Detailed presentation of the different numerical methods used within the domains covered by this book |

***What are the domain's perspectives?***

– Data assimilation is a method undergoing rapid expansion within our field of application. It is increasingly applied within the framework of computing-code calibration and problematic issues encountered in real time. Meteorology was one of the first disciplines to use these methods owing to the large quantity of measurements and observations resulting from work in the field. It has arrived at a level of maturity that means it can now *serve as a reference to other disciplines such as hydrology and hydraulics*.

– These methods will also be used to install measurement systems that are to be increasingly adapted to simulation models. In hydrology, for example, staff gauge stations were installed in areas presenting high stakes, without the entirety of the forecast chain being taken into account. The installation of new models will be accompanied by an approach aimed at optimizing the measurement systems to be assimilated. Likewise, it will be possible for gauging in rivers, very dangerous in the event of a flood, to be considered in relation to hydrodynamic-model usage in order to be able to optimize their installation and enable measurements at maximum reservoir level to be taken in less exposed locations.

# Table of Contents

|  |      |
|--|------|
| <b>Introduction</b> . . . . .  | xiii |
| <b>PART 1. GENERAL CONSIDERATIONS CONCERNING NUMERICAL TOOLS</b> . . .                                 | 1    |
| <b>Chapter 1. Feedback on the Notion of a Model and the Need for Calibration</b> . . . . .             | 3    |
| Denis DARTUS   |      |
| 1.1. “Static” and “dynamic” calibrations of a model . . . . .  | 6    |
| 1.1.1. Static calibration . . . . .  | 6    |
| 1.1.1.1. Static calibration methods . . . . .  | 6    |
| 1.1.1.2. Role of static calibration . . . . .  | 8    |
| 1.1.1.3. Problems associated with static calibration . . . . .   | 9    |
| 1.2. “Dynamic” calibration of a model or data assimilation . . . . .                                   | 10   |
| 1.3. Bibliography . . . . .  | 10   |
| <b>Chapter 2. Engineering Model and Real-Time Model</b> . . . . .                                      | 11   |
| Jean-Michel TANGUY   |      |
| 2.1. Categories of modeling tools . . . . .  | 11   |
| 2.2. Weather forecasting at Météo France . . . . .   | 12   |
| 2.2.1. Objective analysis . . . . .  | 14   |
| 2.2.2. Expertise – publication (output of results) . . . . .   | 16   |
| 2.3. Flood forecasting . . . . .   | 18   |
| 2.4. Characteristics of real-time models . . . . .   | 23   |
| 2.5. Environment of real-time platforms . . . . .  | 25   |
| 2.6. Interpretation of hydrological forecasting by those responsible<br>for civil protection . . . . . | 27   |
| 2.7. Conclusion . . . . .  | 29   |
| 2.8. Bibliography . . . . .  | 30   |

|  |           |
|--|-----------|
| <b>Chapter 3. From Mathematical Model to Numerical Model . . . . .</b>                   | <b>31</b> |
| Jean-Michel TANGUY   |           |
| 3.1. Classification of the systems of differential equations . . . . .                   | 32        |
| 3.2. 3D, 2D, 1D systems . . . . .  | 33        |
| 3.2.1. Reduction in the number of dimensions of the problem . . . . .                    | 33        |
| 3.2.1.1. Two-dimensional horizontal model (2DH model) . . . . .                          | 34        |
| 3.2.1.2. Two-dimensional vertical model (2DV model) . . . . .                            | 36        |
| 3.2.1.3. One-dimensional (longitudinal) model (1D model) . . . . .                       | 37        |
| 3.2.2. Removal of terms from the equations . . . . .                                     | 39        |
| 3.3. Discrete systems and continuous systems . . . . .                                   | 40        |
| 3.4. Equilibrium and propagation problems . . . . .                                      | 41        |
| 3.4.1. Permanent (equilibrium) or boundary value problems . . . . .                      | 41        |
| 3.4.2. Propagation or transitory problems . . . . .                                      | 42        |
| 3.5. Linear and non-linear systems . . . . .   | 43        |
| 3.5.1. Systems of first- and second-order partial differential equations . .             | 45        |
| 3.5.1.1. Introduction to the notion of characteristic . . . . .                          | 45        |
| 3.5.2. Second-order hyperbolic, parabolic and elliptic equations . . . . .               | 46        |
| 3.5.2.1. Hyperbolic problems . . . . .   | 48        |
| 3.5.2.2. Parabolic problems . . . . .  | 50        |
| 3.5.2.3. Elliptic problems . . . . .   | 51        |
| 3.5.3. Applications of the characteristics method . . . . .                              | 52        |
| 3.5.3.1. Additions complementing the method . . . . .                                    | 52        |
| 3.5.3.2. Super-critical and sub-critical flows with<br>Saint-Venant's equation . . . . . | 52        |
| 3.5.3.3. Numerical impacts with the non-linear convection<br>equation . . . . .          | 55        |
| 3.5.3.4. Summary table of the equation types . . . . .                                   | 56        |
| 3.6. Conclusion . . . . .  | 57        |
| 3.7. Bibliography . . . . .  | 57        |
| <b>PART 2. DISCRETIZATION METHODS . . . . .</b>  | <b>59</b> |
| <b>Chapter 4. Problematic Issues Encountered . . . . .</b>                               | <b>61</b> |
| Marie-Madeleine MAUBOURGUET  |           |
| 4.1. Examples of unstable problems . . . . .   | 62        |
| 4.1.1. Pure diffusion equation . . . . .   | 62        |
| 4.1.2. Saint-Venant 2DH equation . . . . .   | 63        |
| 4.2. Loss of material . . . . .  | 63        |
| 4.2.1. Navier-Stokes equations . . . . .   | 63        |
| 4.2.2. Saint-Venant 2DH equation . . . . .   | 65        |
| 4.3. Unsuitable scheme . . . . .   | 66        |
| 4.3.1. Diffusive scheme . . . . .  | 67        |
| 4.4. Bibliography . . . . .  | 69        |

|   |     |
|---|-----|
| <b>Chapter 5. General Presentation of Numerical Methods</b> . . . . .                   | 71  |
| Serge PIPERNO and Alexandre ERN   |     |
| 5.1. Introduction . . . . .   | 71  |
| 5.2. Finite difference method . . . . .   | 72  |
| 5.2.1. Principles of the method . . . . .   | 72  |
| 5.2.2. Essential properties . . . . .   | 74  |
| 5.2.3. Extensions . . . . .   | 75  |
| 5.3. Finite volume method . . . . .   | 77  |
| 5.3.1. Introduction . . . . .   | 77  |
| 5.3.2. Principles of the method . . . . .   | 77  |
| 5.4. Finite element method . . . . .  | 78  |
| 5.4.1. Principles of the method . . . . .   | 79  |
| 5.4.2. Essential properties . . . . .   | 82  |
| 5.4.3. Evolution problems . . . . .   | 86  |
| 5.4.4. Discontinuous finite elements . . . . .  | 88  |
| 5.5. Comparison of the different methods on a convection/diffusion<br>problem . . . . . | 92  |
| 5.6. Bibliography . . . . .   | 93  |
| <b>Chapter 6. Finite Differences</b> . . . . .  | 95  |
| Marie-Madeleine MAUBOURGUET and Jean-Michel TANGUY                                      |     |
| 6.1. General principles of the finite difference method . . . . .                       | 95  |
| 6.2. Discretization of initial and boundary conditions . . . . .                        | 102 |
| 6.2.1. Neumann condition . . . . .  | 103 |
| 6.3. Resolution on a 2D domain . . . . .  | 105 |
| 6.3.1. Summary . . . . .  | 107 |
| <b>Chapter 7. Introduction to the Finite Element Method</b> . . . . .                   | 109 |
| Jean-Michel TANGUY  |     |
| 7.1. Elementary FEM concepts and presentation of the section . . . . .                  | 109 |
| 7.2. Method of approximation by finite elements . . . . .                               | 111 |
| 7.2.1. Definitions . . . . .  | 112 |
| 7.2.2. Rule for partitioning the domain into elements . . . . .                         | 114 |
| 7.3. Geometric transformation . . . . .   | 114 |
| 7.3.1. Notion of a reference element in one dimension . . . . .                         | 114 |
| 7.3.2. Expression using overall coordinates . . . . .                                   | 115 |
| 7.3.3. Expression using local coordinates of the element . . . . .                      | 115 |
| 7.3.4. Expression using local “reference” coordinates . . . . .                         | 115 |
| 7.3.5. 2D approach on a three-node triangular element . . . . .                         | 118 |
| 7.3.6. General approach . . . . .   | 119 |
| 7.4. Transformation of derivation and integration operators . . . . .                   | 121 |
| 7.4.1. Transformation of derivation operators . . . . .                                 | 121 |
| 7.4.2. Expression of the Jacobian matrix $[J]$ and its inverse $[J]^{-1}$ . . . . .     | 123 |

|   |            |
|---|------------|
| 7.4.3. Transformation of an integral . . . . .                                | 125        |
| 7.5. Geometric definition of the elements . . . . .                           | 125        |
| 7.6. Method of weighted residuals . . . . .                                   | 128        |
| 7.7. Transformation of integral forms . . . . .                               | 130        |
| 7.7.1. Integration by parts . . . . .   | 130        |
| 7.7.2. Weak integral form . . . . .   | 131        |
| 7.8. Matrix presentation of the finite element method . . . . .               | 133        |
| 7.8.1. Finite element method . . . . .  | 133        |
| 7.8.2. Discretized elementary integral forms of $W^e$ . . . . .               | 137        |
| 7.8.2.1. Matrix expression of $W^e$ . . . . .                                 | 137        |
| 7.8.2.2. Case of a non-linear operator $L$ . . . . .                          | 140        |
| 7.9. Integral form of $W^e$ on the reference element . . . . .                | 140        |
| 7.9.1. Transformation of derivations . . . . .                                | 140        |
| 7.9.2. Transformation of the integration domain . . . . .                     | 141        |
| 7.9.3. A few conventional forms of $W^e$ and elementary<br>matrices . . . . . | 141        |
| 7.9.4. Assembly of the discretized overall form $W$ . . . . .                 | 144        |
| 7.9.4.1. Overall and elementary variables . . . . .                           | 145        |
| 7.9.4.2. Elementary $\{u_n\}$ and overall $\{U_n\}$ vectors . . . . .         | 145        |
| 7.10. Introduction of the Dirichlet-type boundary conditions . . . . .        | 148        |
| 7.10.1. Dominant diagonal term method . . . . .                               | 148        |
| 7.10.2. Unit term on the diagonal method . . . . .                            | 149        |
| 7.10.3. Equation removal method . . . . .                                     | 150        |
| 7.11. Summary: implementation of the finite element method . . . . .          | 151        |
| 7.12. Application example: wave propagation . . . . .                         | 151        |
| 7.12.1. Berkhoff equations . . . . .  | 152        |
| 7.12.2. Boundary conditions . . . . .   | 153        |
| 7.12.3. Integral formulation . . . . .  | 155        |
| 7.13. Bibliography . . . . .  | 158        |
| <b>Chapter 8. Presentation of the Finite Volume Method . . . . .</b>          | <b>161</b> |
| Alexandre ERN and Serge PIPERNO, section 8.6 written by Dominique THIÉRY      |            |
| 8.1. 1D conservation equations . . . . .                                      | 162        |
| 8.1.1. 1D scalar conservation laws . . . . .                                  | 163        |
| 8.1.2. Systems of 1D conservation laws . . . . .                              | 167        |
| 8.2. Classical, weak and entropic solutions . . . . .                         | 170        |
| 8.2.1. Introduction . . . . .   | 170        |
| 8.2.2. Weak solutions of the conservation equation . . . . .                  | 170        |
| 8.2.3. Entropy conditions, entropic solutions . . . . .                       | 172        |
| 8.3. Numerical solution of a conservation law . . . . .                       | 175        |
| 8.3.1. Finite volume method . . . . .   | 175        |
| 8.3.2. Godunov method . . . . .   | 177        |
| 8.3.3. Examples of Godunov methods . . . . .                                  | 180        |

|   |            |
|---|------------|
| 8.3.4. Complete solution algorithm for the traffic model . . . . .                                  | 181        |
| 8.3.5. Approximate Riemann solvers . . . . .  | 182        |
| 8.4. Numerical solution of hyperbolic systems . . . . .   | 183        |
| 8.4.1. 1D cases . . . . .   | 183        |
| 8.4.2. Approximate Riemann solvers . . . . .  | 187        |
| 8.4.3. 2D finite volume method . . . . .  | 189        |
| 8.4.4. Complete solution algorithm for a two-dimensional problem . . .                              | 192        |
| 8.5. High-order, finite volume methods . . . . .  | 194        |
| 8.6. Application of the finite volume method to the flow development of<br>groundwater . . . . .    | 195        |
| 8.6.1. Confined aquifer with a meshing formed by uniform cubes . . . .                              | 196        |
| 8.6.1.1. Homogeneous aquifer system, no source term, under<br>permanent flow . . . . .              | 199        |
| 8.6.1.2. Aquifer system with no source term, under steady<br>state flow . . . . .                   | 199        |
| 8.6.1.3. Aquifer system with no source term . . . . .   | 200        |
| 8.6.1.4. General case with a source term . . . . .  | 201        |
| 8.6.2. Confined aquifer with a meshing formed by irregular<br>parallelepipeds . . . . .             | 201        |
| 8.6.3. Monolayer unconfined aquifer with a meshing formed by<br>irregular parallelepipeds . . . . . | 202        |
| 8.6.4. Systems of equations and resolution . . . . .  | 203        |
| 8.6.5. Resolution of non-linear systems . . . . .   | 204        |
| 8.6.6. Computing exchange coefficients between two adjacent<br>meshes . . . . .                     | 204        |
| 8.6.7. Taking the boundary conditions into account . . . . .  | 207        |
| 8.6.7.1. Processing an impervious limit . . . . .   | 207        |
| 8.6.7.2. Processing a prescribed head mesh . . . . .  | 207        |
| 8.6.7.3. Introducing an exchange flow onto a limit . . . . .  | 207        |
| 8.6.8. Extending the finite volume method to more complex<br>meshing . . . . .                      | 207        |
| 8.6.8.1. Columns and rows with variable dimensions . . . . .  | 208        |
| 8.6.8.2. Meshes that are no longer parallelepipeds or<br>hexahedrons . . . . .                      | 208        |
| 8.6.8.3. Nested meshings . . . . .  | 208        |
| 8.7. Bibliography . . . . .   | 210        |
| <b>Chapter 9. Spectral Methods in Meteorology . . . . .</b>   | <b>213</b> |
| Jean COIFFIER   |            |
| 9.1. Introduction . . . . .   | 213        |
| 9.2. Using finite series expansion of functions . . . . .   | 214        |
| 9.2.1. General ideas about Galerkin methods . . . . .   | 214        |
| 9.2.2. The various applications of the Galerkin method . . . . .                                    | 215        |

|   |            |
|---|------------|
| 9.3. The spectral method on the sphere . . . . .  | 216        |
| 9.3.1. Historical background . . . . .  | 216        |
| 9.3.2. The basis of surface spherical harmonics . . . . .   | 216        |
| 9.3.3. Properties of the spherical harmonics . . . . .  | 218        |
| 9.3.4. Expansion of a spherical field . . . . .   | 220        |
| 9.3.5. Truncated expansion . . . . .  | 221        |
| 9.3.6. Computing linear terms . . . . .   | 222        |
| 9.3.7. Computing non-linear terms . . . . .   | 223        |
| 9.3.8. Practical implementation of the spectral method . . . . .  | 226        |
| 9.4. The spectral method on a biperiodic domain . . . . .   | 227        |
| 9.4.1. Constructing a biperiodic domain . . . . .   | 227        |
| 9.4.2. The basis functions . . . . .  | 228        |
| 9.4.3. Elliptic truncation . . . . .  | 230        |
| 9.4.4. Computing linear terms . . . . .   | 231        |
| 9.4.5. Computing non-linear terms . . . . .   | 231        |
| 9.4.6. Benefits of the method . . . . .   | 232        |
| 9.5. Bibliography . . . . .   | 232        |
| <b>Chapter 10. Numerical-Scheme Study . . . . .</b>   | <b>235</b> |
| Jean-Michel TANGUY  |            |
| 10.1. Reminder of the notion of the numerical scheme . . . . .  | 235        |
| 10.2. Time discretization . . . . .   | 236        |
| 10.2.1. First-order temporal discretization: semi-implicit scheme . . . . .                             | 236        |
| 10.2.2. Second-order temporal discretization: explicit scheme . . . . .                                 | 237        |
| 10.2.3. Third-order temporal discretization: explicit scheme . . . . .                                  | 238        |
| 10.2.4. First-order temporal discretization: implicit scheme . . . . .                                  | 240        |
| 10.3. Space discretization . . . . .  | 240        |
| 10.4. Scheme study: notions of consistency, stability and convergence . . . . .                         | 241        |
| 10.4.1. Truncation error – consistency . . . . .  | 242        |
| 10.4.2. Stability . . . . .   | 243        |
| 10.4.2.1. Von Neumann method . . . . .  | 244        |
| 10.4.2.2. Applications on the one-dimensional convection<br>equation . . . . .                          | 245        |
| 10.4.2.3. Comment regarding the CFL (Courant, Friedrich<br>and Levy) condition . . . . .                | 250        |
| 10.4.2.4. Modified-equation method . . . . .  | 251        |
| 10.4.2.5. Stability study based on the modified equation . . . . .                                      | 252        |
| 10.4.2.6. Explicit Euler scheme centered with respect to space . . . . .                                | 253        |
| 10.4.2.7. Lax-Wendroff scheme (2nd order) . . . . .   | 255        |
| 10.4.2.8. Behavior of the 2nd-order LW scheme (LW2), applied<br>to the 1D convection equation . . . . . | 255        |
| 10.4.2.9. Implicit Euler scheme . . . . .   | 256        |
| 10.4.2.10. Matrix method . . . . .  | 258        |

|   |            |
|---|------------|
| 10.4.3. Convergence . . . . .   | 260        |
| 10.4.4. Example: study of a numerical scheme applied to a PDE . . . . .                   | 262        |
| 10.4.4.1. Summary table of the properties of the schemes studied . . . . .                | 262        |
| 10.5. Bibliography . . . . .  | 264        |
| <b>Chapter 11. Resolution Methods . . . . .</b>   | <b>267</b> |
| Marie-Madeleine MAUBOURGUET   |            |
| 11.1. Temporal integration methods . . . . .  | 268        |
| 11.2. Linearization methods for non-linear systems . . . . .                              | 270        |
| 11.3. Methods for solving linear systems $AX = B$ . . . . .                               | 271        |
| 11.3.1. Direct methods . . . . .  | 271        |
| 11.3.2. Iterative methods . . . . .   | 271        |
| 11.4. Bibliography . . . . .  | 272        |
| <b>PART 3. INTRODUCTION TO DATA ASSIMILATION . . . . .</b>                                | <b>273</b> |
| <b>Chapter 12. Data Assimilation . . . . .</b>  | <b>275</b> |
| Jean PAILLEUX, Denis DARTUS, Xijun LAI, Jérôme MONNIER<br>and Marc HONNORAT               |            |
| 12.1. Several examples of the application of data assimilation . . . . .                  | 277        |
| 12.1.1. Data assimilation in meteorology . . . . .  | 277        |
| 12.1.2. Data assimilation in hydrology . . . . .  | 280        |
| 12.1.2.1. Global sensitivity analysis . . . . .   | 281        |
| 12.1.2.2. Temporal sensitivity analysis . . . . .   | 281        |
| 12.1.2.3. Spatial sensitivity analysis . . . . .  | 282        |
| 12.1.2.4. Identification of the Richards parameters . . . . .                             | 282        |
| 12.2. Data assimilation in hydraulics with the Dassflow model . . . . .                   | 284        |
| 12.2.1. Example of the Pearl River . . . . .  | 287        |
| 12.3. Bibliography . . . . .  | 290        |
| <b>Chapter 13. Data Assimilation Methodology . . . . .</b>                                | <b>295</b> |
| Hélène BESSIÈRE, Hélène ROUX, François-Xavier LE DIMET<br>and Denis DARTUS                |            |
| 13.1. Representation of the system . . . . .  | 295        |
| 13.2. Taking errors into account . . . . .  | 296        |
| 13.3. Simplified approach to optimum static estimation theory . . . . .                   | 297        |
| 13.3.1. First approach: minimization of the variance in the estimation<br>error . . . . . | 298        |
| 13.3.2. Second approach: weighted least squares . . . . .                                 | 299        |
| 13.4. Generalization in the multidimensional case . . . . .                               | 300        |
| 13.4.1. Minimization of the variance of the linear estimator with<br>background . . . . . | 301        |
| 13.4.2. Weighted least squares . . . . .  | 302        |

|   |            |
|---|------------|
| 13.5. The different data assimilation techniques . . . . .                  | 303        |
| 13.6. Sequential assimilation method: the Kalman filter . . . . .           | 304        |
| 13.7. Extension to non-linear models: the extended Kalman filter . . . . .  | 307        |
| 13.8. Assessment of the Kalman filter . . . . .                             | 308        |
| 13.9. Variational methods . . . . .   | 312        |
| 13.10. Discreet formulation of the cost function: the 3D-VAR . . . . .      | 313        |
| 13.11. General variational formalism: the 4D-VAR . . . . .                  | 314        |
| 13.12. Continuous formulation of the cost function . . . . .                | 314        |
| 13.12.1. The adjoint method . . . . .                                       | 316        |
| 13.13. Principle of automatic differentiation . . . . .                     | 322        |
| 13.14. Summary of variational methods . . . . .                             | 322        |
| 13.15. A complete application example: the Burgers equation . . . . .       | 324        |
| 13.15.1. Analytical resolution using the adjoint method . . . . .           | 325        |
| 13.15.2. Using automatic differentiation . . . . .                          | 331        |
| 13.16. Feedback on the notion of a model and the need for calibration . . . | 335        |
| 13.16.1. Modeling guidelines, adapted from Schlesinger . . . . .            | 336        |
| 13.16.2. Static calibration of a model . . . . .                            | 339        |
| 13.16.2.1. Static calibration methods . . . . .                             | 339        |
| 13.16.2.2. Role of static calibration . . . . .                             | 341        |
| 13.16.2.3. Problems associated with static calibration . . . . .            | 342        |
| 13.16.3. “Dynamic” calibration of a model or data assimilation . . . . .    | 343        |
| 13.17. Bibliography . . . . .   | 343        |
| <b>List of Authors . . . . .</b>  | <b>349</b> |
| <b>Index . . . . .</b>  | <b>351</b> |
| <b>General Index of Authors . . . . .</b>                                   | <b>353</b> |
| <b>Summary of the Other Volumes in the Series . . . . .</b>                 | <b>355</b> |