# Computational Continuum Mechanics

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## COMPUTATIONAL CONTINUUM MECHANICS

This book presents the nonlinear theory of continuum mechanics and demonstrates its use in developing nonlinear computer formulations for large displacement dynamic analysis. Basic concepts used in continuum mechanics are presented and used to develop nonlinear general finite element formulations that can be effectively used in large displacement analysis. The book considers two nonlinear finite element dynamic formulations: a general large-deformation finite element formulation and then a formulation that can efficiently solve small deformation problems that characterize very and moderately stiff structures. The book presents material clearly and systematically, assuming the reader has only basic knowledge in matrix and vector algebra and dynamics. The book is designed for use by advanced undergraduates and first-year graduate students. It is also a reference for researchers, practicing engineers, and scientists working in computational mechanics, bio-mechanics, computational biology, multibody system dynamics, and other fields of science and engineering using the general continuum mechanics theory.

Ahmed A. Shabana is the Richard and Loan Hill Professor of Engineering at the University of Illinois, Chicago. Dr. Shabana received his PhD in mechanical engineering from the University of Iowa. His active areas of research interest are in dynamics, vibration, and control of mechanical systems. He is also the author of other books, including *Theory of Vibration: An Introduction, Vibration of Discrete and Continuous Systems, Computational Dynamics, Railroad Vehicle Dynamics: A Computational Approach*, and *Dynamics of Multibody Systems, Third Edition.*

# Preface

Nonlinear continuum mechanics is one of the fundamental subjects that form the foundation of modern computational mechanics. The study of the motion and behavior of materials under different loading conditions requires understanding of basic, general, and nonlinear, kinematic and dynamic relationships that are covered in continuum mechanics courses. The finite element method, on the other hand, has emerged as a powerful tool for solving many problems in engineering and physics. The finite element method became a popular and widely used computational approach because of its versatility and generality in solving large-scale and complex physics and engineering problems. Nonetheless, the success of using the continuum-mechanics-based finite element method in the analysis of the motion of bodies that experience general displacements, including arbitrary large rotations, has been limited. The solution to this problem requires resorting to some of the basic concepts in continuum mechanics and putting the emphasis on developing sound formulations that satisfy the principles of mechanics. Some researchers, however, have tried to solve fundamental formulation problems using numerical techniques that lead to approximations. Although numerical methods are an integral part of modern computational algorithms and can be effectively used in some applications to obtain efficient and accurate solutions, it is the opinion of many researchers that numerical methods should only be used as a last resort to fix formulation problems. Sound formulations must be first developed and tested to make sure that these formulations satisfy the basic principles of mechanics. The equations that result from the use of the analytically correct formulations can then be solved using numerical methods.

This book is focused on presenting the nonlinear theory of continuum mechanics and demonstrating its use in developing nonlinear computer formulations that can be used in the large displacement dynamic analysis. To this end, the basic concepts used in continuum mechanics are first presented and then used to develop nonlinear general finite element formulations that can be effectively used in the large displacement analysis. Two nonlinear finite element dynamic formulations will be considered in this book. The first is a general large-deformation finite element formulation, whereas the second is a formulation that can be used efficiently to solve small-deformation problems that characterize very and moderately stiff structures.

In this latter case, an elaborate method for eliminating the unnecessary degrees of freedom must be used in order to be able to efficiently obtain a numerical solution. An attempt has been made to present the materials in a clear and systematic manner with the assumption that the reader has only basic knowledge in matrix and vector algebra as well as basic knowledge of dynamics. The book is designed for a course at the senior undergraduate and first-year graduate level. It can also be used as a reference for researchers and practicing engineers and scientists who are working in the areas of computational mechanics, biomechanics, computational biology, multibody system dynamics, and other fields of science and engineering that are based on the general continuum mechanics theory.

In **Chapter 1** of this book, matrix, vector, and tensor notations are introduced. These notations will be repeatedly used in all chapters of the book, and, therefore, it is necessary that the reader reviews this chapter in order to be able to follow the presentation in subsequent chapters. The polar decomposition theorem, which is fundamental in continuum and computational mechanics, is also presented in this chapter. D'Alembert's principle and the principle of virtual work can be used to systematically derive the equations of motion of physical systems. These two important principles are discussed, and the relationship between them is explained. The use of a finite dimensional model to describe the continuum motion is also discussed in Section 8; whereas in Section 9, the procedure for developing the discrete equations of motion is outlined. In Section 10, the principles of momentum and principle of work and energy are presented. In this section, the problems associated with some of the finite element formulations that violate these analytical mechanics principles are discussed. Section 11 of Chapter 1 is devoted to a discussion on the definitions of the gradient vectors that are used in continuum mechanics to define the strain components.

In **Chapter 2**, the general kinematic displacement equations of a continuum are developed. These equations are used to define the strain components. The Green–Lagrange strains and the Almansi or Eulerian strains are introduced. The Green–Lagrange strains are defined in the reference configuration, whereas the Almansi or Eulerian strains are defined in the current deformed configuration. The relationships between these strain components are established and used to shed light on the physical meaning of the strain components. Other deformation measures as well as the velocity and acceleration equations are also defined in this chapter. The important issue of objectivity that must be considered when large deformations and inelastic formulations are used is discussed. The equations that govern the change of volume and area, the conservation of mass, and examples of deformation modes are also presented in this chapter.

Forces and stresses are discussed in **Chapter 3**. Equilibrium of forces acting on an infinitesimal material element is used to define the Cauchy stresses, which are used to develop the partial differential equations of equilibrium. The transformation of the stress components and the symmetry of the Cauchy stress tensor are among the topics discussed in this chapter. The virtual work of the forces due to the change of the shape of the continuum is defined. The deviatoric stresses, stress objectivity, and energy balance equations are also discussed in Chapter 3.

The definition of the strain and stress components is not sufficient to describe the motion of a continuum. One must define the relationship between the stresses and strains using the constitutive equations that are discussed in **Chapter 4**. In Chapter 4, the generalized Hooke's law is introduced, and the assumptions used in the definition of homogeneous isotropic materials are outlined. The principal strain invariants and special large-deformation material models are discussed. The linear and nonlinear viscoelastic material behavior is also discussed in Chapter 4.

In many engineering applications, plastic deformations occur due to excessive forces and impact as well as thermal loads. Several plasticity formulations are presented in **Chapter 5**. First, a one-dimensional theory is used in order to discuss the main concepts and solution procedures used in the plasticity analysis. The theory is then generalized to the three-dimensional analysis for the case of small strains. Large strain nonlinear plasticity formulations as well as the $J_2$ flow theory are among the topics discussed in Chapter 5. This chapter can be skipped in its entirety because it has no effect on the continuity of the presentation, and the developments in subsequent chapters do not depend on the theory of plasticity in particular.

Nonlinear finite element formulations are discussed in Chapter 6 and 7. Two formulations are discussed in these two chapters. The first is a large-deformation finite element formulation, which is discussed in **Chapter 6**. This formulation, called the absolute nodal coordinate formulation, is based on a continuum mechanics theory and employs displacement gradients as coordinates. It leads to a unique displacement and rotation fields and imposes no restrictions on the amount of rotation or deformation within the finite element. The absolute nodal coordinate formulation has some unique features that distinguish it from other existing large-deformation finite element formulations: it leads to a constant mass matrix; it leads to zero centrifugal and Coriolis forces; it automatically satisfies the principles of mechanics; it correctly describes an arbitrary rigid-body motion including finite rotations; and it can be used to develop several beams, plate, and shell elements that relax many of the assumptions used in classical theorems because this formulation allows for the use of more general constitutive relationships.

Clearly, large-deformation finite element formulations can also be used to solve small deformation problems. However, it is not recommended to use a large-deformation finite element formulation to solve a small-deformation problem. Large-deformation formulations do not exploit some particular features of small-deformation problems, and, therefore, such formulations can be very inefficient in the solution of stiff and moderately stiff systems. It turns out that the development of an efficient small-deformation finite element formulation that correctly describes an arbitrary rigid-body motion requires the use of more elaborate techniques in order to define a local linear problem without compromising the ability of the method to describe large-displacement small-deformation behavior. The finite element floating frame of reference formulation, which is widely used in the analysis of small deformations, is discussed in **Chapter 7** of this book. This formulation allows eliminating high-frequency modes that do not have a significant effect on the solution, thereby leading to a lower-dimension dynamic model that can be efficiently solved using numerical and computer methods.

I would like to thank many students and colleagues with whom I worked for several years on the subject of flexible body dynamics. I was fortunate to collaborate with excellent students and colleagues who educated me in this important field of computational mechanics. In particular, I would like to thank two of my doctorate students, Bassam Hussein and Luis Maqueda, who provided solutions for several of the examples presented in Chapter 4 and Chapter 5. I am grateful for the help I received from Mr. Peter Gordon, the Engineering Editor, and the production staff of Cambridge University Press. It was a pleasant experience working with them on the production of this book. I would also like to thank my family for their help, patience, and understanding during the time of preparing this book.

Ahmed A. Shabana
Chicago, IL, 2007

# Contents

# 1 INTRODUCTION

Matrix, vector, and tensor algebras are often used in the theory of continuum mechanics in order to have a simpler and more tractable presentation of the subject. In this chapter, the mathematical preliminaries required to understand the matrix, vector, and tensor operations used repeatedly in this book are presented. Principles of mechanics and approximation methods that represent the basis for the formulation of the kinematic and dynamic equations developed in this book are also reviewed in this chapter. In the first two sections of this chapter, matrix and vector notations are introduced and some of their important identities are presented. Some of the vector and matrix results are presented without proofs because it is assumed that the reader has some familiarity with matrix and vector notations. In Section 3, the summation convention, which is widely used in continuum mechanics texts, is introduced. This introduction is made despite the fact that the summation convention is rarely used in this book. Tensor notations, on the other hand, are frequently used in this book and, for this reason, tensors are discussed in Section 4. In Section 5, the *polar decomposition theorem*, which is fundamental in continuum mechanics, is presented. This theorem states that any nonsingular square matrix can be decomposed as the product of an orthogonal matrix and a symmetric matrix. Other matrix decompositions that are used in computational mechanics are also discussed. In Section 6, D'Alembert's principle is introduced, while Section 7 discusses the virtual work principle. The finite element method is often used to obtain finite dimensional models of continuous systems that in reality have infinite number of degrees of freedom. To introduce the reader to some of the basic concepts used to obtain finite dimensional models, discussions of approximation methods are included in Section 8. The procedure for developing the discrete equations of motion is outlined in Section 9, while the principle of conservation of momentum and the principle of work and energy are discussed in Section 10. In continuum mechanics, the gradients of the position vectors can be determined by differentiation with respect to different parameters. The change of parameters can lead to the definitions of strain components in different directions. This change of parameters, however, does not change the coordinate system in which the gradient vectors are defined. The effect of the change of parameters on the definitions of the gradients is discussed in Section 11.

## 1.1 MATRICES

In this section, some identities, results, and properties from matrix algebra that are used repeatedly in this book are presented. Some proofs are omitted, with the assumption that the reader is familiar with the subject of linear algebra.

**Definitions** An $m \times n$ matrix $\mathbf{A}$ is an ordered rectangular array, which can be written in the following form:

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1.1}$$

where $a_{ij}$ is the $ij$th element that lies in the $i$th row and $j$th column of the matrix. Therefore, the first subscript $i$ refers to the row number, and the second subscript $j$ refers to the column number. The arrangement of Equation 1 shows that the matrix $\mathbf{A}$ has $m$ rows and $n$ columns. If $m = n$, the matrix is said to be *square*, otherwise the matrix is said to be *rectangular*. The *transpose* of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix, denoted as $\mathbf{A}^{\mathrm{T}}$, which is obtained from $\mathbf{A}$ by exchanging the rows and columns, that is $\mathbf{A}^{\mathrm{T}} = (a_{ji})$.

A *diagonal matrix* is a square matrix whose only nonzero elements are the diagonal elements, that is, $a_{ij} = 0$ if $i \neq j$. An *identity* or *unit matrix*, denoted as $\mathbf{I}$, is a diagonal matrix that has all its diagonal elements equal to one. The *null* or *zero matrix* is a matrix that has all its elements equal to zero. The *trace* of a square matrix $\mathbf{A}$ is the sum of all its diagonal elements, that is,

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} \tag{1.2}$$

This equation shows that $\mathrm{tr}(\mathbf{I}) = n$, where $\mathbf{I}$ is the identity matrix and $n$ is the dimension of the matrix.

A square matrix $\mathbf{A}$ is said to be *symmetric* if

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}, \quad a_{ij} = a_{ji} \tag{1.3}$$

A square matrix is said to be *skew symmetric* if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}, \quad a_{ij} = -a_{ji} \tag{1.4}$$

This equation shows that all the diagonal elements of a skew-symmetric matrix must be equal to zero. That is, if $\mathbf{A}$ is a skew-symmetric matrix with dimension $n$, then $a_{ii} = 0$ for $i = 1, 2, \dots, n$. Any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix. For example, if **B** is a square matrix, **B** can be written as

$$\mathbf{B} = \bar{\mathbf{B}} + \tilde{\mathbf{B}} \tag{1.5}$$

where $\bar{\mathbf{B}}$ and $\tilde{\mathbf{B}}$ are, respectively, symmetric and skew-symmetric matrices defined as

$$\bar{\mathbf{B}} = \frac{1}{2}\left(\mathbf{B} + \mathbf{B}^{\mathrm{T}}\right), \quad \tilde{\mathbf{B}} = \frac{1}{2}\left(\mathbf{B} - \mathbf{B}^{\mathrm{T}}\right) \tag{1.6}$$

Skew-symmetric matrices are used in continuum mechanics to characterize the rotations of the material elements.

**Determinant** The *determinant* of an $n \times n$ square matrix **A**, denoted as $|\mathbf{A}|$ or $\det(\mathbf{A})$, is a scalar quantity. In order to be able to define the unique value of the determinant, some basic definitions have to be introduced. The *minor* $M_{ij}$ corresponding to the element $a_{ij}$ is the determinant of a matrix obtained by deleting the $i$th row and $j$th column from the original matrix **A**. The *cofactor* $C_{ij}$ of the element $a_{ij}$ is defined as

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{1.7}$$

Using this definition, the determinant of the matrix **A** can be obtained in terms of the cofactors of the elements of an arbitrary row $j$ as follows:

$$|\mathbf{A}| = \sum_{k=1}^{n} a_{jk} C_{jk} \tag{1.8}$$

One can show that the determinant of a diagonal matrix is equal to the product of the diagonal elements, and the determinant of a matrix is equal to the determinant of its transpose; that is, if **A** is a square matrix, then $|\mathbf{A}| = \left|\mathbf{A}^{\mathrm{T}}\right|$. Furthermore, the interchange of any two columns or rows only changes the sign of the determinant. It can also be shown that if the matrix has linearly dependent rows or linearly dependent columns, the determinant is equal to zero. A matrix whose determinant is equal to zero is called a *singular matrix*. For an arbitrary square matrix, singular or nonsingular, it can be shown that the value of the determinant does not change if any row or column is added or subtracted from another.

It can be shown that the determinant of the product of two matrices is equal to the product of their determinants. That is, if **A** and **B** are two square matrices, then $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

As will be shown in this book, the determinants of some of the deformation measures used in continuum mechanics are used in the formulation of the energy expressions. Furthermore, the relationship between the volume of a continuum in the undeformed state and the deformed state is expressed in terms of the determinant of the matrix of position vector gradients. Therefore, if the elements of a square matrix depend on a parameter, it is important to be able to determine the derivatives of the determinant with respect to this parameter. Using Equation 8, one can show that if the elements of the matrix **A** depend on a parameter $t$, then

$$\frac{d}{dt}|\mathbf{A}| = \sum_{k=1}^{n} \dot{a}_{1k}C_{1k} + \sum_{k=1}^{n} \dot{a}_{2k}C_{2k} + \ldots + \sum_{k=1}^{n} \dot{a}_{nk}C_{nk} \tag{1.9}$$

where $\dot{a}_{ij} = da_{ij}/dt$. The use of this equation is demonstrated by the following example.

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**EXAMPLE 1.1**

Consider the matrix **J** defined as

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

where $J_{ij} = \partial r_i/\partial x_j$, and **r** and **x** are the vectors

$$\mathbf{r}(x_1, x_2, x_3, t) = [r_1 \quad r_2 \quad r_3]^{\mathrm{T}}, \quad \mathbf{x} = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$$

That is, the elements of the vector **r** are functions of the coordinates $x_1$,$x_2$, and $x_3$ and the parameter $t$. If $J = |\mathbf{J}|$ is the determinant of **J**, prove that

$$\frac{dJ}{dt} = \left(\frac{\partial \dot{r}_1}{\partial r_1} + \frac{\partial \dot{r}_2}{\partial r_2} + \frac{\partial \dot{r}_3}{\partial r_3}\right) J$$

where $\partial \dot{r}_i/\partial r_j = \left(\partial/\partial r_j\right)(dr_i/dt)$, $i, j = 1, 2, 3$.

*Solution:* Using Equation 9, one can write

$$\frac{dJ}{dt} = \sum_{k=1}^{3} \dot{J}_{1k}C_{1k} + \sum_{k=1}^{3} \dot{J}_{2k}C_{2k} + \sum_{k=1}^{3} \dot{J}_{3k}C_{3k}$$

where $C_{ij}$ is the cofactor associated with element $J_{ij}$. Note that the preceding equation can be written as

$$\frac{dJ}{dt} = \begin{vmatrix} \dot{J}_{11} & \dot{J}_{12} & \dot{J}_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ \dot{J}_{21} & \dot{J}_{22} & \dot{J}_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ \dot{J}_{31} & \dot{J}_{32} & \dot{J}_{33} \end{vmatrix}$$

In this equation,

$$\dot{J}_{ij} = \frac{\partial \dot{r}_i}{\partial x_j} = \frac{\partial \dot{r}_i}{\partial r_1}\frac{\partial r_1}{\partial x_j} + \frac{\partial \dot{r}_i}{\partial r_2}\frac{\partial r_2}{\partial x_j} + \frac{\partial \dot{r}_i}{\partial r_3}\frac{\partial r_3}{\partial x_j} = \sum_{k=1}^{3} \frac{\partial \dot{r}_i}{\partial r_k} J_{kj}$$

Using this expansion, one can show that

$$\begin{vmatrix} \dot{J}_{11} & \dot{J}_{12} & \dot{J}_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_1}{\partial r_1}\right) J$$

Similarly, one can show that

$$\begin{vmatrix} J_{11} & J_{12} & J_{13} \\ \dot{J}_{21} & \dot{J}_{22} & \dot{J}_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_2}{\partial r_2}\right) J, \quad \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ \dot{J}_{31} & \dot{J}_{32} & \dot{J}_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_3}{\partial r_3}\right) J$$

Using the preceding equations, it is clear that

$$\frac{dJ}{dt} = \left(\frac{\partial \dot{r}_1}{\partial r_1} + \frac{\partial \dot{r}_2}{\partial r_2} + \frac{\partial \dot{r}_3}{\partial r_3}\right) J$$

This matrix identity is important and is used in this book to evaluate the rate of change of the determinant of the matrix of position vector gradients in terms of important deformation measures.

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**Inverse and Orthogonality** A square matrix $\mathbf{A}^{-1}$ that satisfies the relationship

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{1.10}$$

where $\mathbf{I}$ is the identity matrix, is called the *inverse* of the matrix $\mathbf{A}$. The inverse of the matrix $\mathbf{A}$ is defined as

$$\mathbf{A}^{-1} = \frac{\mathbf{C}_t}{|\mathbf{A}|} \tag{1.11}$$

where $\mathbf{C}_t$ is the *adjoint* of the matrix $\mathbf{A}$. The adjoint matrix $\mathbf{C}_t$ is the transpose of the matrix of the cofactors $(C_{ij})$ of the matrix $\mathbf{A}$. One can show that the determinant of the inverse $|\mathbf{A}^{-1}|$ is equal to $1/|\mathbf{A}|$.

A square matrix is said to be *orthogonal* if

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I} \tag{1.12}$$

Note that in the case of an orthogonal matrix $\mathbf{A}$, one has

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1} \tag{1.13}$$

That is, the inverse of an orthogonal matrix is equal to its transpose. One can also show that if $\mathbf{A}$ is an orthogonal matrix, then $|\mathbf{A}| = \pm 1$; and if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two orthogonal matrices that have the same dimensions, then their product $\mathbf{A}_1\mathbf{A}_2$ is also an orthogonal matrix.

Examples of orthogonal matrices are the $3 \times 3$ transformation matrices that define the orientation of coordinate systems. In the case of a right-handed coordinate system, one can show that the determinant of the transformation matrix is +1; this is a *proper orthogonal transformation*. If the right-hand rule is not followed, the determinant of the resulting orthogonal transformation is equal to –1, which is an *improper orthogonal transformation*, such as in the case of a reflection.

**Matrix Operations** The sum of two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined as

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}) \tag{1.14}$$

In order to add two matrices, they must have the same dimensions. That is, the two matrices $\mathbf{A}$ and $\mathbf{B}$ must have the same number of rows and same number of columns in order to apply Equation 14.

The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is another matrix $\mathbf{C}$ defined as

$$\mathbf{C} = \mathbf{AB} \tag{1.15}$$

The element $c_{ij}$ of the matrix $\mathbf{C}$ is defined by multiplying the elements of the $i$th row in $\mathbf{A}$ by the elements of the $j$th column in $\mathbf{B}$ according to the rule

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_k a_{ik}b_{kj} \tag{1.16}$$

Therefore, the number of columns in $\mathbf{A}$ must be equal to the number of rows in $\mathbf{B}$. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then $\mathbf{C}$ is an $m \times p$ matrix. In general, $\mathbf{AB} \neq \mathbf{BA}$. That is, matrix multiplication is not commutative. The *associative law* for matrix multiplication, however, is valid; that is, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$, provided consistent dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are used.

## 1.2 VECTORS

Vectors can be considered special cases of matrices. An $n$-dimensional vector $\mathbf{a}$ can be written as

$$\mathbf{a} = (a_i) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1 \quad a_2 \quad \ldots \quad a_n]^{\mathrm{T}} \tag{1.17}$$