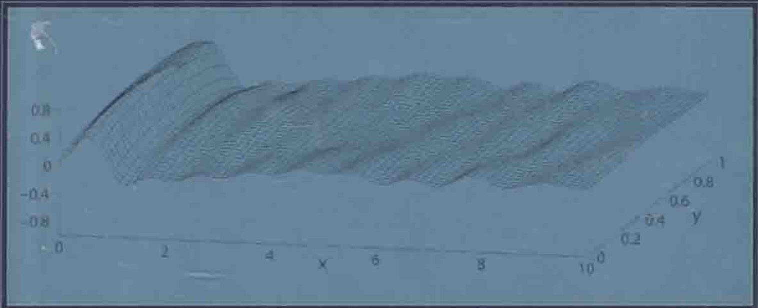
Fourier Series and Numerical Methods *for* Partial Differential Equations



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FOURIER SERIES AND NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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FOURIER SERIES AND NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

Dodi, Ben, and Hannah

PREFACE

The importance of partial differential equations in modeling phenomena in engineering and the physical, natural, and social sciences is known by many students and practitioners in these fields. I came to know this in my studies of atmospheric science. A dependent variable, such as air temperature, is generally a function of two or more independent variables (time and three spatial coordinates). Our desire to understand and predict the state of natural systems, such as our atmosphere, frequently begins with equations involving rates of change (partial derivatives) of these quantities with respect to the independent variables. Once these equations have evolved from the conceptual models of such systems, the next challenge is "solving" these partial differential equations in qualitative and quantitative ways.

This book on solution techniques for partial differential equations has evolved over the last six years and several offerings of an introductory course on partial differential equations at Luther College. Students enter the course with a background typical of most junior- or senior-level mathematics or physical science majors including two to three calculus courses, an introductory linear algebra course, and a one-semester course in ordinary differential equations. With this common foundation, the book intends to strengthen and extend the reader's knowledge and appreciation of the power of linear spaces and linear transformations for purposes of understanding and solving a wide range of equations including many important partial differential equations. The notions of infinite dimensional vector spaces, scalar product, and norm lead to, perhaps, the reader's initial introduction to Hilbert spaces through the theoretical development of Fourier series and properties of convergence. These somewhat abstract foundations are important aspects of developing an undergraduate's more general problem solving skill.

Most "real-world" partial differential equations, because of their nonlinear nature, do not lend themselves to solution techniques of separation of variables, orthogonal eigenfunction bases, and Fourier series solutions. Consequently, three different numerical solution techniques are introduced in the final third of the book. The versatile finite difference method is introduced first because of its relative understandable and easy implementation. The finite element method is a popular method used by many sanctioners in a variety of fields. Yet, it has a formidable theoretical foundation including concepts of infinite-dimensional function spaces and finite-dimensional subspaces. The third method for numerical solutions is the finite analytic method wherein separation of variables Fourier series methods are applied to locally linearized versions of the original partial differential equation.

Admittedly, I do not cover all of this material in a one-semester course. Usually, Chapters 1 – 5 are covered, and then topics are chosen from Chapters 6 and 7. Chapter 9 on finite differences is covered, and then either an introduction to finite elements or the finite analytic method completes the semester.

Because Maple© is our campus computer algebra system of choice, a "library" of Maple work sheets has been developed over the years. They are useful for solving many of the exercises ranging from one-dimensional problems using Fourier series to multidimensional problems using the various numerical techniques. The work sheets are available for users of the book through the textbook web site: http://faculty.luther.edu/ bernatzr/PDE Text/index.html

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Decorah, Iowa
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R.A.B.

CONTENTS

CHAPTER 1

INTRODUCTION

This chapter introduces general topics concerning partial differential equations (PDEs). It begins with basic terminology associated with PDEs and then describes how PDEs are classified. The PDEs common to scientific and engineering fields are introduced. Next, the notion of initial and boundary value problems is introduced. The chapter ends with a brief discussion of various solution techniques, with an introductory example of the method of separation of variables. This example also serves as motivation for developing the material on Fourier series in Chapter 2.

1.1 TERMINOLOGY AND NOTATION

Suppose u is a function of the spatial variable x and time t so that $u = u(x,t)$. Recall that the **partial derivative** of u with respect to x is defined as

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h} \tag{1.1}$$

Fourier Series and Numerical Methods for Partial Differential Equations, First Edition. By Richard Bernatz

The partial derivative of u with respect to t is defined in a similar way. Another way of representing the partial of u with respect to x is

$$\frac{\partial u}{\partial x} = u_x$$

A partial differential equation is an equation involving one or more partial derivatives of a dependent variable u. An example of such is the one-dimensional (1D) diffusion equation

$$u_t = k u_{xx} \tag{1.2}$$

where u_t represents the first partial of u with respect to t, k is a constant diffusion coefficient, and u_{xx} represents the **second partial** of u with respect to x. That is,

$$u_{xx} = \lim_{h \to 0} \frac{u_x(x+h,t) - u_x(x,t)}{h} \tag{1.3}$$

Some applications of PDEs may include a **mixed partial**, such as u_{xy}, where

$$u_{xy} = \lim_{h \to 0} \frac{u_x(x,y+h,t) - u_x(x,y,t)}{h} \tag{1.4}$$

with $u = u(x, y, t)$.

It is common for the dependent variable u to be a function of three spatial variables x, y, and z, as well as time t. The **general form** of a PDE for u in this case may be expressed as

$$F(x, y, z, t; u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{yx}, u_{yy}, \ldots, u_{tt}, \ldots) = 0 \tag{1.5}$$

1.2 CLASSIFICATION

The **order** of a PDE is the highest order derivative in Equation (1.5). The order of the most common PDEs in science and engineering applications is two or less. In the event that $u = u(x, y)$, Equation (1.5) may be expressed as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = Q \tag{1.6}$$

The PDE is said to be **linear** if each coefficient $A - Q$ is at most a function of x or y. Otherwise, the equation is **nonlinear**. A nonlinear equation is **quasilinear** if it is linear in its highest order derivatives. Examples of PDEs with identification of their order and linearity are given in Table 1.1. If the term Q on the right-hand side of Equation (1.6) is zero, the PDE is said to be **homogeneous**. Otherwise the PDE is classified as **nonhomogeneous**.

Suppose Equation (1.6) is linear. All equations of this form may be further classified as **parabolic**, **hyperbolic**, or **elliptic**. Heat flow and diffusion problems are typically described by **parabolic** forms of Equation (1.6). These equations have coefficients A, B, and C satisfying the property $B^2 - 4AC = 0$. **Hyperbolic** forms of

Table 1.1 PDEs: Order and Linearity.

PDE	Order	Linearity
$u_y u_x + \alpha u_y = 2$	1st	nonlinear
$u_{xx} + \alpha u u_y = 0$	1st	quasilinear
$x u_x + \alpha u_y = u^2 + 1$	1st	quasilinear
$x u_x + \alpha u_y = 1$	1st	linear
$u_{xx} + u_{yy} = \cos(x^2 + y)$	2nd	linear
$u u_{xx} + \alpha u_y^2 = 0$	2nd	nonlinear
$u_{xx} + u u_y = 0$	2nd	quasilinear

the equation are those for which $B^2 - 4AC > 0$. Common equations of this type are associated with vibrating systems and wave motion. Finally, when $B^2 - 4AC < 0$, the equation is **elliptic**. Equations of this type typically represent steady-state (time independent) phenomena.

If any of the coefficients A, B, or C are functions of x or y, the characterization of the equation as parabolic, hyperbolic, or elliptic may be a function of location in the xy-plane. As an example, consider the PDE

$$y^2 u_{xx} + \sqrt{x} u_{xy} + u_{yy} + 2u_x = 0 \tag{1.7}$$

The expression $B^2 - 4AC$ is equal to $x - 4y^2$ for Equation (1.7). Consequently, the equation is parabolic on the curve $x = 4y^2$, hyperbolic for points (x, y) such that $x > 4y^2$, and elliptic for ordered pairs (x, y) that satisfy $4y^2 > x$ and $x \geq 0$. These regions are depicted in Figure 1.1.

1.3 CANONICAL FORMS

For the case where the dependent variable u is a function of at most two independent variables, as indicated in Equation 1.6, there are three common types of PDEs. The Laplace equation is $u_{xx} + u_{yy} = 0$, and is elliptic on the entire xy-plane. The heat equation has the form $u_t - u_{xx} = 0$, where the independent variable t represents time and x represents a spatial dimension. The heat equation is parabolic on the entire xy-plane. The wave equation $u_{tt} - u_{xx} = 0$ is hyperbolic on the entire xy-plane. Here, the dependent variable u represents the displacement or wave height as a function of time t and location x.

It can be shown that with a smooth, nonsingular change of coordinates, the sign of the discriminant $B^2 - 4AC$ will not change. Therefore, it is possible to make a change of coordinate transformation in which an elliptic PDE is transformed to a Laplace equation, a parabolic PDE is transformed to the heat equation, and a hyperbolic equation is transformed into the wave equation form. Consequently, the Laplace,