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VOLUME X

COMBINATORIAL ANALYSIS

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PREFACE

Problems in combinatorial analysis range from the study of finite geometries, through algebra and number theory, to the domains of communication theory and transportation networks. Although the questions that arise are all problems of arrangement, they differ enormously in the superficial form in which they arise, and quite often intrinsically, as well.

Perhaps the greatest discrepancy is between the discrete problems involving the construction of designs and the continuous problems of linear inequalities. Nevertheless, in a number of the papers that are presented a basic unity of the whole theory is brought to light. For example, Alan Hoffman has shown that many problems of discrete choice and arrangement may be solved in an elegant fashion by means of recent developments of the theory of linear inequalities, a continuation of work of Dantzig and Fulkerson. Similarly, Robert Kalaba and Richard Bellman have shown that a variety of combinatorial problems arising in the study of scheduling and transportation can be treated by means of functional equation techniques. Marshall Hall has observed that the solution of a problem in arrangements, in particular, the construction of pairs of orthogonal squares, is precisely equivalent to solving a certain equation for a matrix with nonnegative real entries.

A very challenging area of research which is investigated in a number of the papers that follow is that of using a computer to attack combinatorial questions, both by means of theoretical algorithms and by means of sophisticated search techniques. Papers by Paige and Tompkins, Walker, Gerstenhaber, Flood, Gleason, Lehmer, Swift, Todd, and Gomory, discuss versions of this fundamental problem.

Following the manner in which the Symposium was divided into four sessions, the Proceedings are divided into four sections. These are:

I. Existence and construction of combinatorial designs.
II. Combinatorial analysis of discrete extremal problems.
III. Problems of communications, transportation and logistics.
IV. Numerical analysis of discrete problems.

What is very attractive about this field of research is that it combines both the most abstract and most nonquantitative parts of mathematics with the most arithmetic and numerical aspects. It shows very clearly that the discovery of a feasible solution of a particular problem may necessitate enormous theoretical advances. Perhaps the moral of the tale is that the division into pure and applied mathematics is certainly artificial and to the detriment of the enthusiasts on both sides. Furthermore, the way in which apparently simple problems require a complex medley of algebraic, geometric, analytic and numerical considerations shows that the traditional subdivisions of mathematics are themselves too rigidly labelled. There is one subject, mathematics, and one type of problem, a mathematical problem.

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CONTENTS

CURRENT STUDIES ON COMBINATORIAL DESIGNS

BY

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1. **Introduction.** Current Studies on Combinatorial Designs have taken us into Number Theory, Group Theory, Matrix Theory, and to a certain extent into the theory of convex bodies. Number Theory and Group Theory have been used almost from the beginning of the theory of Combinatorial Analysis, but the more recent uses have been of a different nature.

The nature of some of the earlier methods may be illustrated in the theory of Steiner triple systems. A Steiner triple system is an arrangement of n objects into triples in such a way that every pair of distinct objects occurs in exactly one triple. It is trivial that n must be of the form $6k + 1$ or $6k + 3$. In 1859, six years after Steiner [**21**] had posed the problem, Reiss [**20**] showed that systems exist for every such value. Reiss's method was a recursively constructive method. By a fairly complicated construction he showed how, given a system with $t > 1$ objects he could construct one with $2t + 1$ objects and another with $2t - 5$. Starting with $t = 3$ we may obtain all possible values $6k + 1$ and $6k + 3$ recursively. A more general recursive method of constructing Steiner triple systems is due to E. H. Moore [**18**] who proved the theorem.

THEOREM. *If there is a Steiner triple system of order t_2 containing a subsystem of order t_3, and if there is also a Steiner triple system of order $t_1 > 1$, then we can construct a system of order $t = t_3 + t_1(t_2 - t_3)$.*

If $6k + 1 = p$ is a prime and r is a primitive root of p, then the sets of residues mod p $i, i + r^a, i + r^{a+k}$, $a = 0, \cdots, k - 1$, $i = 0, \cdots, 6k$ may be shown to be a Steiner triple system. Steiner triple systems also admit a composition, since if there are systems of orders t_1 and t_2 there is also a system of order t_1t_2. This is easy to see in the following way: Given a Steiner triple system S, let us construct a quasi-group Q from the elements of S by the rules (1) $a^2 = a$ and (2) if $b \neq a$ and a, b, c is the triple of S containing a, b, put $ab = c$. Q may be characterized by the properties $a^2 = a$, $(ab)b = a$. Then the direct product of two such quasi-groups has the same property and this yields the composition rule.

In problems of enumeration, the theory of generating functions has been used from the beginning, and is still of great value, particularly in the study of problems of partitions. But I shall not concern myself here with this branch of Combinatorial Analysis. The symbolic calculus so extensively developed by MacMahon has been successful in giving formal algebraic equivalents of many combinatorial problems, but I cannot think of any recent instance in which this approach has given either practical methods for constructing designs or for proving theorems about them.

For the greatest part, I shall speak of block designs. A *block design* is an arrangement of v objects into b blocks, each consisting of k distinct objects such that each object occurs in exactly r blocks and each pair of objects occurs in exactly λ blocks. The two following conditions on the five parameters are elementary:

$$bk = vr, \quad r(k-1) = \lambda(v-1). \tag{1.1}$$

A Steiner triple system is a block design with $k = 3$, $\lambda = 1$. A *symmetric block design* satisfies the further condition $v = b$, whence also $k = r$. In a symmetric block design the value $k - \lambda = n$ plays a central arithmetical role. A symmetric block design with $\lambda = 1$ is a *finite projective plane*, its parameters being

$$\begin{aligned} v &= b = n^2 + n + 1, \\ r &= k = n + 1, \\ \lambda &= 1. \end{aligned} \tag{1.2}$$

Here n is said to be the *order* of the plane.

2. **Matrices and quadratic forms.** Let $a_1, \cdots, a_v$ be the objects of a block design D and $B_1, \cdots, B_b$ the blocks. Let us define incidence numbers a_{ij}, $i = 1, \cdots, v$, $j = 1, \cdots, b$, where $a_{ij} = 1$ if $a_i \in B_j$ and $a_{ij} = 0$ if $a_i \notin B_j$. Then the design is fully described by the $v \times b$ *incidence matrix*,

$$A = (a_{ij}), \quad i = 1, \cdots, v, \quad j = 1, \cdots, b. \tag{2.1}$$

If A^T is the transpose of A, then the defining properties of the design imply

$$AA^T = B = (r - \lambda)I + \lambda S, \tag{2.2}$$

where I is the $v \times v$ identity matrix and S is a $v \times v$ matrix consisting entirely of 1's. It is easy to evaluate the determinant of B.

$$\det B = (r - \lambda)^{v-1}(r + (v-1)\lambda). \tag{2.3}$$

If $r = \lambda$ then the design is the trivial one in which every block contains all the objects. Otherwise $r > \lambda$ and B is non-singular. Since the rank of B cannot exceed the rank of A, we have

$$b \geqq v, \tag{2.4}$$

an inequality first proved by R. A. Fisher [8] by other means. Furthermore if D is a symmetric design $b = v$, $k = r$ and conditions (1.1) reduce to

$$k(k-1) = \lambda(v-1). \tag{2.5}$$

In this case $k + (v-1)\lambda = k^2$, and of course

$$\det B = (\det A)^2. \tag{2.6}$$

Here (2.3) and (2.6) show immediately that the following theorem holds:

THEOREM 2.1. *In a symmetric block design, if v is even then $n = k - \lambda$ must be a square.*

For a symmetric design we find that the incidence matrix A is *normal*, i.e.

$$A^T A = AA^T = B = nI + \lambda S. \tag{2.7}$$

This expresses a duality in the design in particular the fact that in a symmetric design any two distinct blocks have exactly λ objects in common.

The matrix equation (2.2) has an alternate representation in terms of quadratic forms. Let $x_1, \cdots, x_v$ be indeterminates and let us use the incidence numbers a_{ij} to define linear forms

$$L_j = \sum_{i=1}^{v} a_{ij} x_i. \tag{2.8}$$

Then (2.3) is equivalent to

$$\begin{aligned} L_1^2 + L_2^2 + \cdots + L_b^2 &= (r - \lambda)(x_1^2 + \cdots + x_v^2) + \lambda(x_1 + \cdots + x_v)^2 \\ &= Q(x_1, \cdots, x_v). \end{aligned} \tag{2.9}$$

For a symmetric design (2.9) takes the form

$$\begin{aligned} L_1^2 + L_2^2 + \cdots + L_v^2 &= n(x_1^2 + \cdots + x_v^2) + \lambda(x_1 + \cdots + x_v)^2 \\ &= Q(x_1, \cdots, x_v). \end{aligned} \tag{2.10}$$

A major step in the study of Combinatorial Analysis was taken by Bruck and Ryser [4] in 1949 when they introduced the matrix notation given here and reasoned as follows: In (2.10) the linear forms L_j have rational coefficients (indeed the integers 0 and 1) and hence (2.10) gives a rational representation of the quadratic form Q by the form $L_1^2 + \cdots + L_v^2$. Thus the deep Hasse-Minkowski criteria for the rational equivalence of quadratic forms are applicable. Using this technique for finite projective planes, they proved the following result:

THEOREM 2.2. *A necessary condition for the existence of a finite projective plane with $n + 1$ points on a line is that if $n \equiv 1, 2 \pmod 4$ then $n = a^2 + b^2$ for appropriate integers a and b.*

This shows that finite projective planes do not exist for an infinite set of values of n beginning with $n = 6, 14, 21, 22, \cdots$. This result is in sharp contrast to the results on Steiner triple systems, where every value of the parameters consistent with the basic relations (1.1) is possible. The value $n = 6$ had previously been shown impossible by Tarry [22] by straightforward enumeration.

Theorem 2.2 was generalized by Chowla and Ryser [6] to symmetric block designs.

THEOREM 2.3. *If a symmetric design exists with parameters v, k, λ and $n = k - \lambda$, then* (1) *for v even, n is a square and* (2) *for v odd, the equation*

$$z^2 = nx^2 + (-1)^{(v-1)/2}\lambda y^2$$

is solvable in integers not all zero.

A different use was made of equation (2.9) by W. S. Connor [7]. If we specify the first t blocks of a design we have determined the first t forms $L_1, \cdots, L_t$ of (2.9). Then we have

$$L_{t+1}^2 + \cdots + L_b^2 = Q(x_1, \cdots, x_v) - L_1^2 - \cdots - L_t^2 = Q^*. \tag{2.11}$$

If there exists a design with these initial blocks, then $L_{t+1}, \cdots, L_b$ exist and in particular the form Q^* must be positive semi-definite. Hence a necessary condition for the existence of a design with t specified blocks is that Q^* be positive semi-definite. Connor gives a test from this property in terms of a determinant. Let the t given blocks be $B_1, \cdots, B_t$ and let s_{ij} be the number of objects common to B_i and B_j for $i, j = 1, \cdots, t$. Form the matrix

$$\begin{aligned} &C_t = (c_{ij}), \quad i, j = 1, \cdots, t \\ &c_{ii} = (r - k)(r - \lambda), \quad c_{ij} = \lambda k - r s_{ij}, \quad i \neq j. \end{aligned} \tag{2.12}$$

Then the determinant of C_t must satisfy

$$\begin{aligned} &\text{(i)} && \det |C_t| \geqq 0 \quad \text{if} \quad t < b - v, \\ &\text{(ii)} && \det |C_t| = 0 \quad \text{if} \quad t > b - v, \\ &\text{(iii)} && k(r)^{-b+v+1}(r - \lambda)^{2v-b-1} \det C_{b-v} \end{aligned} \tag{2.13}$$

is a perfect integral square if there is to exist a design with blocks $B_1, \cdots, B_t$. As one consequence he finds inequalities for the s_{ij}. He shows

$$\frac{1}{r}[2\lambda k + r(r - \lambda - k)] \geqq s_{ij} \geqq -r + k + \lambda. \tag{2.14}$$

For the symmetric designs we have $r = k$ and (2.14) gives $s_{ij} = \lambda$, the result previously noted, being equivalent to the duality of the symmetric designs and the normality of the incidence matrix A in (2.7). One of the applications of this method was the proof of the following embedding theorem by Hall and Connor [**11**].

THEOREM 2.4. *A block design with parameters $v = 2^{-1}t(t + 1)$, $b = 2^{-1}(t + 1)(t + 2)$, $r = t + 2$, $k = t$, $\lambda = 2$ can be embedded in a symmetric block design with $v = b = 2^{-1}(t^2 + 3t + 4)$, $r = k = t + 2$, $\lambda = 2$.*

This is the analogue, with $\lambda = 2$ instead of $\lambda = 1$ of the well known result that an affine plane can be embedded in a projective plane by adjoining a line at infinity. An example due to Bhattacharya shows the corresponding theorem for $\lambda = 3$ to be false. Connor and others have applied his method to the study of partially balanced designs and other generalizations of block designs.

If we have t blocks $B_1, \cdots, B_t$ as the initial blocks of a symmetric design and if the obviously necessary condition $s_{ij} = \lambda$ holds, then the matrix C_t of (2.12) is identically zero and the Connor method gives no further information. This seemed a little strange and Hall and Ryser [**13**] endeavored to find out more about this situation. The results obtained indicate that indeed in the real field and more strongly, even in the rational field, no further information is available beyond that of Theorem 2.3 and the condition $s_{ij} = \lambda$. The precise state of affairs is given by the following theorem:

THEOREM 2.5. (NORMAL COMPLETION THEOREM). *Let v, k, λ be integers such that $k(k-1) = \lambda(v-1)$ and such that the conditions of Theorem 2.3 are satisfied. Let A_t be a rational $v \times t$ matrix whose columns all sum to k, and such that $A_t^T A_t = (k-\lambda)I_t + \lambda S_t$, I_t and S_t being $t \times t$ matrices such that I_t is the identity and S_t consisting entirely of 1's. Then there exists a rational $v \times v$ matrix A which has A_t for its first t columns such that $A^T A = AA^T = (k-\lambda)I + \lambda S$.*

Note that the hypothesis on A_t is certainly satisfied if this is the matrix of blocks $B_1, \cdots, B_t$ for which $s_{ij} = \lambda$. This says that not only is there a rational matrix A completing A_t to give a solution of the quadratic condition (2.9) but even a normal matrix A satisfying $A^T A = AA^T$. One corollary of this theorem is that the existence of a rational matrix X satisfying

$$X^T X = (k-\lambda)I + \lambda S$$

implies the existence of a rational A satisfying

$$A^T A = AA^T = (k-\lambda)I + \lambda S.$$

A special case of this last result had been proved previously by Albert [**1**].

3. **A problem in convex spaces.** Suppose we are given two superposed $n \times n$ orthogonal squares. For $n = 4$ an example is:

11	22	33	44
23	14	41	32
34	43	12	21
42	31	24	13

In general we have an $n \times n$ square and each cell contains a first and a second digit, chosen from 1 to n.

The first digits and second digits separately are Latin squares, i.e. each of 1 to n occurs exactly once in each row and once in each column. Furthermore the squares are *orthogonal*, i.e. in the superposed square the pairs of first and second digits occurring are all the combinations 11, 12, $\cdots$, nn. A pair of orthogonal squares may be regarded as representing four parallel pencils in a finite affine plane with n^2 points. For let each of the n^2 cells be associated with a point.

In the first pencil let there be n lines, the ith containing the points of the ith row of the square. In the second let there be n lines each containing the points, the jth containing the points of the jth column. For the third pencil let the kth line, $k = 1, \cdots, n$ consist of those points whose cells have k as their first digit. For the fourth pencil let the tth line, $t = 1, \cdots, n$ consist of those points whose cells have t as their second digit. Then geometrically we have n^2 points and four sets of n lines such that

(1) Each line contains n points.

(2) Two lines of the same family are parallel.

(3) Two lines of different families have exactly one point in common.

(4) Through each of the n^2 points there is exactly one line of each family.

For property (3) as respects the third and fourth pencils, this is the orthogonality condition. Our construction may easily be reversed so that from four parallel pencils satisfying (1), (2), (3), (4) we may construct a pair of orthogonal $n \times n$ Latin squares. Thus orthogonal squares are not only interesting in themselves, but their existence is a necessary condition for the existence of an affine plane (and so also projective plane) of order n whenever $n \geqq 3$. Euler conjectured that orthogonal squares do not exist if $n \equiv 2 \pmod 4$. Tarry [22] verified this for $n = 6$ by trial, but up to the present no theorem on this exists[1] and the attempt to test $n = 10$ will be discussed by Tompkins and Paige at this Symposium. Mann [17] has shown that for $n \not\equiv 2 \pmod 4$ two orthogonal squares exist. Thus there exist two orthogonal 21×21 squares although there is no plane of order 21.

Here I formulate the existence problem in terms of real quadratic forms in a way that leads to a number of problems on convex spaces.

Let us take $4n$ variables associating n with each pencil $x_i, i = 1, \cdots, n$ with rows; $y_j, j = 1, \cdots, n$ with columns; $z_k, k = 1, \cdots, n$ with first digits and $w_t, t = 1, \cdots, n$ with second digits. With the rth point $P_r, r = 1, \cdots, n^2$ associate the linear form

$$L_r = x_i + y_j + z_k + w_t \tag{3.1}$$

if P_r is on the ith, jth, kth, tth lines of the respective pencils. Then

$$\begin{aligned} L_1^2 + L_2^2 + \cdots + L_{n^2}^2 &= Q \\ &= n(x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 + z_1^2 + \cdots + z_n^2 + w_1^2 + \cdots + w_n^2) \\ &\quad + 2\sum_{i,j} x_i y_j + 2\sum_{i,k} x_i z_k + 2\sum_{i,t} x_i w_t + 2\sum_{j,k} y_j z_k + 2\sum_{j,t} y_j w_t \\ &\quad + 2\sum_{k,t} z_k w_t. \end{aligned} \tag{3.2}$$

[1] *Note added in proof.* R. C. Bose, S. S. Shrikhande and E. T. Parker have succeeded in constructing pairs of $n \times n$ orthogonal squares for all $n \equiv 2 \pmod 4$, $n \geqq 10$.

The value of Q is easily determined from the defining properties of the pencils. Conversely if $L_1, \cdots, L_{n^2}$ are a selection of n^2 of the n^4 forms of (3.1) satisfying (3.2) then they determine a pair of $n \times n$ orthogonal squares. (For $n = 10$ we have only to choose 100 of 10,000 linear forms.)

THEOREM 3.1. *If for the Q of* (3.2) *we have*

$$Q = U_1^2 + \cdots + U_M^2 \tag{3.3}$$

where the U_m, $m = 1, \cdots, M$ are linear forms in the x, y, z, w with non-negative coefficients, then each of $U_1, \cdots, U_m$ is a scalar multiple of one of the n^4 forms of (3.1).

Proof. In (3.3) each U_m is of the form (3.4) $U_m = a_m x_{i_m} + b_m y_{j_m} + c_m z_{k_m} + d_m w_{t_m}$, where a_m, b_m, c_m, d_m are non-negative, since if a U_m contained as many as two x's (or y's, z's, w's) with positive coefficients, this would give a positive cross product in U_m^2 involving say $x_r x_s$, $r \neq s$ which cannot be canceled by the remaining U^2's and yet is not present in Q. Of course a U_m might conceivably contain no x with a positive coefficient. Now consider those m's for which $i_m = r$, $j_m = s$ and $a_m > 0$, $b_m > 0$. Call $T_{r,s}$ this set of m's. Then, these being precisely those U_m's for which U_m^2 gives a positive term in x_r, y_s, we have

$$\begin{aligned} \sum a_m b_m &= 1, \ m \in T_{r,s}, \\ \sum a_m^2 &= A_{r,s}, \ m \in T_{r,s}, \\ \sum b_m^2 &= B_{r,s}, \ m \in T_{r,s}. \end{aligned} \tag{3.5}$$

Now as

$$\sum (a_m - b_m)^2 \geqq 0 \ m \in T_{r,s} \tag{3.6}$$

we have

$$A_{r,s} - 2 + B_{r,s} \geqq 0 \tag{3.7}$$

or

$$A_{r,s} + B_{r,s} \geqq 2 \tag{3.8}$$

with a strict inequality unless every term in (3.6) is 0.

$$A_r = \sum_{s=1}^{n} A_{r,s} \leqq n \tag{3.9}$$

since the left hand side is $\sum a_m^2$ for all values of m for which $x_{i_m} = x_r$ and there is a y term. Further there is a strict inequality if for any m we have $x_{i_m} = x_r$ but no y term. Similarly

$$B_s = \sum_{r=1}^{n} B_{r,s} \leqq n. \tag{3.10}$$

Combining (3.9) and (3.10)

$$\sum_{r,s} (A_{r,s} + B_{r,s}) \leqq 2n^2. \tag{3.11}$$

But from (3.8)

$$\sum_{r,s} A_{r,s} + B_{r,s} \geqq 2n^2. \tag{3.12}$$

Hence in all of (3.6)$\cdots$(3.12) we must have strict equalities. In particular $a_m = b_m$ for $m \in T_{r,s}$ and for every U with a positive x term there is a positive y term. Continuing we conclude that

$$a_m = b_m = c_m = d_m, \qquad m = 1, \cdots, M \tag{3.13}$$

proving the theorem.

We may say even more about (3.3).

THEOREM 3.2. *In* (3.3) $M \geqq n^2$, *and if* $M = n^2$ *then* $U_1, \cdots, U_M$ *determine a pair of orthogonal* $n \times n$ *Latin squares.*

Proof. Let $U_m = a_m(x_{i_m} + y_{j_m} + z_{k_m} + w_{t_m})$. Each U gives exactly 6 non-zero cross products xy etc. As Q has $6n^2$ non-zero cross products we must have $M \geqq n^2$. If $M = n^2$ then each of the $6n^2$ cross products xy etc. must occur exactly once. Here from U_m we have the cross product $2a_m^2 x_{i_m} y_{j_m} = 2x_{i_m} y_{j_m}$ whence $a_m = 1$ in every case, and the U's are L's and so yield orthogonal squares.

These theorems can be given a matrix formulation:

THEOREM 3.3. *Let* A *be a* $4n \times n^2$ *matrix satisfying*

$$AA^T = \begin{pmatrix} nI & S & S & S \\ S & nI & S & S \\ S & S & nI & S \\ S & S & S & nI \end{pmatrix}.$$

If A *is non-negative then* A *is the incidence matrix for a pair of orthogonal Latin squares.*

The existence problem considered here has aspects relevant to general problems in the theory of convex spaces. The space S of semi-definite quadratic forms is a convex cone. So also is the space P of non-negative quadratic forms. The quadratic forms arising in our combinatorial problems are sums of squares of non-negative linear forms and this is again a convex cone D. Clearly $D \subseteq S \cap P$, but it has been shown by Horn that for 5 or more variables D is a proper subspace of $S \cap P$. Each of S and P is self-adjoint and the adjoint space of $S \cap P$ is $S \cup P$. The adjoint space of D is the space N of quadratic forms non-negative for non-negative arguments. Here $N \supseteq S \cup P$ and the inclusion is proper for forms with 5 or more variables by Horn's result. Let us note that $Q = n^{-2} \sum L^2$, L ranging over the n^4 forms of (3.1). Theorem 3.1 says that a representative of Q as a linear combination of extreme points of D is in fact a linear combination of extreme points of a polyhedron, whose vertices are the squares of the n^4 forms of (3.1). Q/n^2 is indeed the center of gravity of these points. Theorem 3.2 points out that the number of extreme points needed in the representation of Q is vital to our problem. What is the nature of the points given as combination of a limited number of extreme points? In a square the combinations of two extreme points are the edges and diagonals.

4. Group theory and designs. From the beginning one way of constructing designs has been to take a group and find a design which has this as an automorphism group (or collineation group using geometric terminology). Occasionally the elements of the group itself form a design. Thus, given an elementary Abelian group G of order 2^r, let us delete from G the identity. Then the triples of elements of the form a, b, ab (i.e. a subgroup of order 4 with the identity deleted) form a Steiner triple S system on $2^r - 1$ elements. Here S has as its automorphism group not G itself, but $A(G)$ the group of automorphisms of G. As is well known, $A(G)$ is doubly transitive on S and is of order $(2^r - 1)(2^r - 2)\cdots(2^r - 2^{r-1})$.

A result which may now be regarded as classical in the theory of projective planes states that the existence of certain families of configurations is equivalent to the existence of certain collineations. Specifically the validity of the theorem of Desargues for all configurations with a given center and axis is equivalent to the existence of all perspective collineations with that center and axis. The analogue of this state of affairs has not been sufficiently developed for designs in general. Let me however give one theorem of this kind.

THEOREM 4.1. *The following two conditions are equivalent in a Steiner triple system S:*

(1) *For every object $a \in S$ there is an involution α_a of S which fixes a and interchanges an object b with c if a, b, c are a triple of S.*

(2) *Every pair of intersecting triples of S lies in a subsystem which is an S_9, i.e. a Steiner triple system with 9 objects.*

Proof. Let us show that (2) implies (1). Let us designate an object of S as 1. We wish to show that α_1, the permutation fixing 1 and interchanging i with j if 1, i, j are a triple of S is a collineation of S. Clearly, α_1 maps onto themselves all triples including 1. It remains only to show that a triple not including 1 is also mapped onto a triple of S by α_1. Let such a triple be say 2, 4, 6 and let 1, 2, 3 be the triple including 1 and 2. Then by our hypothesis 1, 2, 3 and 2, 4, 6 lie in an S_9. We readily see that this must include

$$\begin{array}{llllll} & 1\ 2\ 3 & \quad 2\ 4\ 6 \\ (4.1) & 1\ 4\ 5 & \\ & 1\ 6\ 7 & \end{array}$$

The S_9 is now easily completed and seen to be of the form

$$(4.2)\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \\ 1 & 8 & 9 \end{array} \qquad \begin{array}{ccc} 2 & 4 & 6 \\ 2 & 5 & 8 \\ 2 & 7 & 9 \\ & & \end{array} \qquad \begin{array}{ccc} 3 & 4 & 9 \\ 3 & 5 & 7 \\ 3 & 6 & 8 \\ & & \end{array} \qquad \begin{array}{ccc} 4 & 7 & 8 \\ 5 & 6 & 9 \\ & & \\ & & \end{array}$$

But then $\alpha_1 = (1)\,(2,3)\,(4,5)\,(6,7)\,(8,9)$ and $(2,4,6)\alpha_1 = 3,5,7$ which is a triple. Thus property (2) implies property (1).

On the other hand let us assume property (1) and let 1,2,3; 1,4,5 be two intersecting triples of S. Then we certainly have triples

$$(4.3)\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \end{array} \qquad \begin{array}{ccc} 2 & 4 & 6 \\ & & \\ & & \end{array}$$

and $\alpha_1 = (1)\,(2,3)\,(4,5)\,(6,7)$. Here $(2,4,6)\alpha_1 = 3,5,7$ must be a triple of S. But then the third element x of $2,5,x$ must be different from $1,\cdots,7$, say 8, and we have 2,5,8 and also a triple 1,8,9. Here $\alpha_1 = (1)\,(2,3)\,(4,5)\,(6,7)\,(8,9)$ and $(2,5,8)\alpha_1 = 3,4,9$. Hence we have triples

$$(4.4)\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \\ 1 & 8 & 9 \end{array} \qquad \begin{array}{ccc} 2 & 4 & 6 \\ 2 & 5 & 8 \\ & & \\ & & \end{array} \qquad \begin{array}{ccc} 3 & 5 & 7 \\ 3 & 4 & 9 \\ & & \\ & & \end{array}$$

and collineations

$$(4.5)\qquad \begin{aligned} \alpha_1 &= (1)\,(2,3)\,(4,5)\,(6,7)\,(8,9) \\ \alpha_2 &= (2)\,(1,3)\,(4,6)\,(5,8) \\ \alpha_3 &= (3)\,(1,2)\,(4,9)\,(5,7) \\ \alpha_4 &= (4)\,(1,5)\,(2,6)\,(3,9) \\ \alpha_5 &= (5)\,(1,4)\,(2,8)\,(3,7) \end{aligned}$$

here

$$(4.6)\qquad \begin{aligned} (1,4,5)\alpha_2 &= 3,6,8 \\ (1,4,5)\alpha_3 &= 2,9,7 \\ (1,2,3)\alpha_4 &= 5,6,9 \\ (1,2,3)\alpha_5 &= 4,8,7 \end{aligned}$$

giving us the complete S_9 containing 1,2,3 and 1,4,5 as above in (4.2). Thus property (1) implies (2).

A number of recent results are to the effect that certain hypotheses imply that a finite plane is Desarguesian. I shall content myself with listing several of these:

Gleason [**9**]. *Every finite Fano plane is Desarguesian.*

Here a Fano plane is a plane in which the diagonal points of a complete quadrilateral are collinear.

Gleason [**9**]. *A finite plane is Desarguesian if for every pair P,l where P is a point lying on the line l there is a non-identical elation with center P and axis l.*