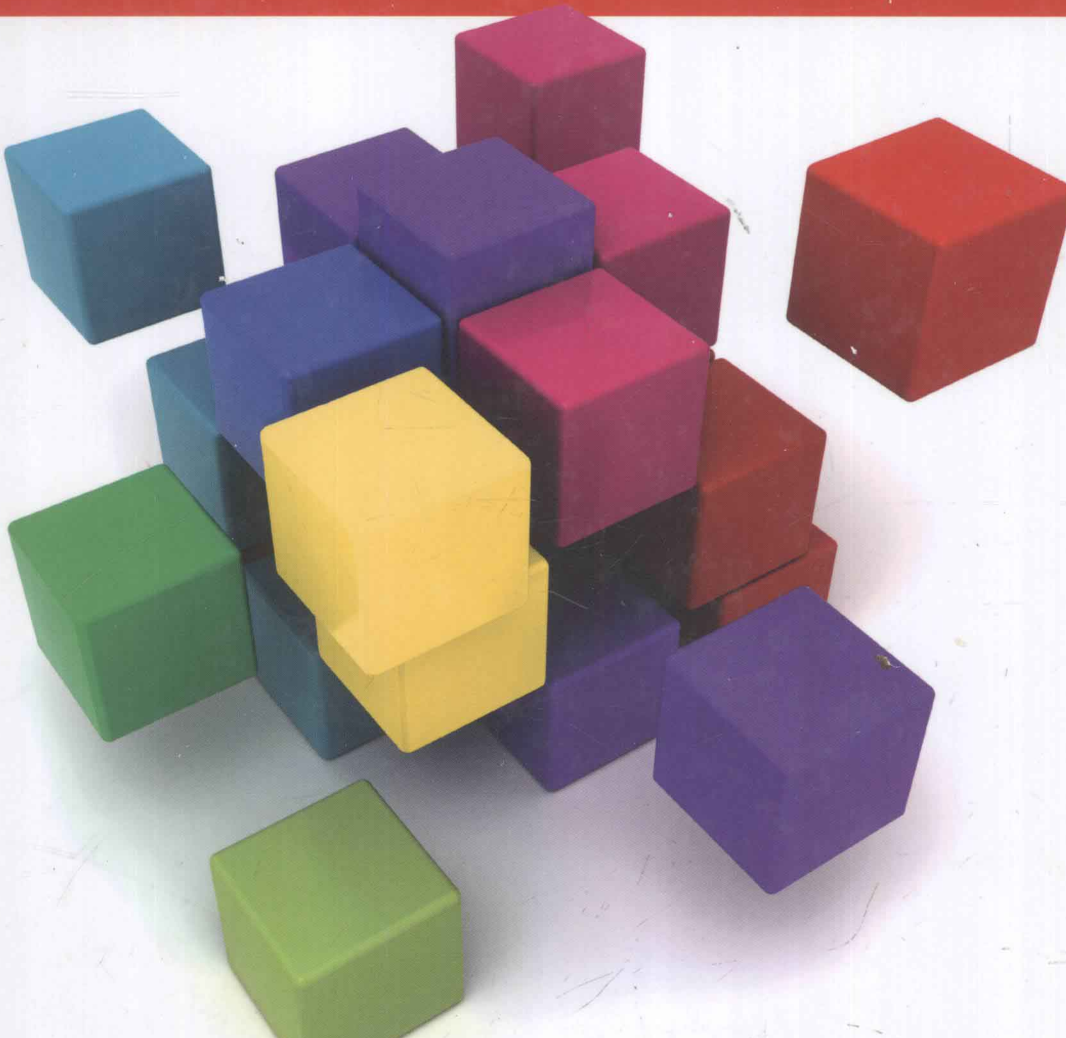
ESSENTIAL MATHEMATICAL METHODS

for the Physical Sciences



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Essential Mathematical Methods for the Physical Sciences

The mathematical methods that physical scientists need for solving substantial problems in their fields of study are set out clearly and simply in this tutorial-style textbook. Students will develop problem-solving skills through hundreds of worked examples, self-test questions and homework problems. Each chapter concludes with a summary of the main procedures and results and all assumed prior knowledge is summarized in one of the appendices. Over 300 worked examples show how to use the techniques and around 100 self-test questions in the footnotes act as checkpoints to build student confidence. Nearly 400 end-of-chapter problems combine ideas from the chapter to reinforce the concepts. Hints and outline answers to the odd-numbered problems are given at the end of each chapter, with fully worked solutions to these problems given in the accompanying *Student Solution Manual*. Fully worked solutions to all problems, password-protected for instructors, are available at www.cambridge.org/essential.

K. F. Riley read mathematics at the University of Cambridge and proceeded to a Ph.D. there in theoretical and experimental nuclear physics. He became a Research Associate in elementary particle physics at Brookhaven, and then, having taken up a lectureship at the Cavendish Laboratory, Cambridge, continued this research at the Rutherford Laboratory and Stanford; in particular he was involved in the experimental discovery of a number of the early baryonic resonances. As well as having been Senior Tutor at Clare College, where he has taught physics and mathematics for over 40 years, he has served on many committees concerned with the teaching and examining of these subjects at all levels of tertiary and undergraduate education. He is also one of the authors of *200 Puzzling Physics Problems* (Cambridge University Press, 2001).

M. P. Hobson read natural sciences at the University of Cambridge, specializing in theoretical physics, and remained at the Cavendish Laboratory to complete a Ph.D. in the physics of star formation. As a Research Fellow at Trinity Hall, Cambridge, and subsequently an Advanced Fellow of the Particle Physics and Astronomy Research Council, he developed an interest in cosmology, and in particular in the study of fluctuations in the cosmic microwave background. He was involved in the first detection of these fluctuations using a ground-based interferometer. Currently a University Reader at the Cavendish Laboratory, his research interests include both theoretical and observational aspects of cosmology, and he is the principal author of *General Relativity: An Introduction for Physicists* (Cambridge University Press, 2006). He is also a Director of Studies in Natural Sciences at Trinity Hall and enjoys an active role in the teaching of undergraduate physics and mathematics.

Preface

Since *Mathematical Methods for Physics and Engineering* (Cambridge: Cambridge University Press, 1998) by Riley, Hobson and Bence, hereafter denoted by *MMPE*, was first published, the range of material it covers has increased with each subsequent edition (2002 and 2006). Most of the additions have been in the form of introductory material covering polynomial equations, partial fractions, binomial expansions, coordinate geometry and a variety of basic methods of proof, though the third edition of *MMPE* also extended the range, but not the general level, of the areas to which the methods developed in the book could be applied. Recent feedback suggests that still further adjustments would be beneficial. In so far as content is concerned, the inclusion of some additional introductory material such as powers, logarithms, the sinusoidal and exponential functions, inequalities and the handling of physical dimensions, would make the starting level of the book better match that of some of its readers.

To incorporate these changes, and others to increase the user-friendliness of the text, into the current third edition of *MMPE* would inevitably produce a text that would be too ponderous for many students, to say nothing of the problems the physical production and transportation of such a large volume would entail. It is also the case that for students for whom a course on mathematical methods is *not* their first engagement with mathematics beyond high school level, all of the additional introductory material, as well as some of that presented in the early chapters of the original *MMPE*, would be ground already covered. For such students, typically those who have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations, much of the first half of such an omnibus edition would be redundant.

For these reasons, we present under the current title, *Essential Mathematical Methods for the Physical Sciences*, an alternative edition of *MMPE*, one that focuses on the core of a putative extended third edition, omitting, except in summary form, all of the "mathematical tools" at one end, and some of the more specialized topics that appear in the third edition at the other. The emphasis is very much on developing the *methods* required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields.

For the record, we note that the more advanced topics in the third edition of *MMPE* that have fallen victim to this approach are quantum operators, tensors, group and representation theory, and numerical methods. The chapters on special functions, and the applications of complex variables have both been reduced to some extent, as have those on probability and statistics.

At the other end of the spectrum, the excised introductory material has not been altogether lost. Indeed, Appendix A of the present text consists entirely of summaries, in the style described in the penultimate paragraph of this Preface, of the material that is presumed to have been previously studied and mastered by the student. Clearly it can be used both as a reference/reminder and as an indicator of some missing background knowledge.

One aspect that has remained constant throughout the three editions of *MMPE*, is the general style of presentation of a topic – a qualitative introduction, physically based wherever possible, followed by a more formal presentation or proof, and finished with one or two full-worked examples. This format has been well received by reviewers, and there is no reason to depart from its basic structure.

In terms of style, many physical science students appear to be more comfortable with presentations that contain significant amounts of verbal explanation and comment, rather than with a series of mathematical equations the last line of which implies "job done". We have made changes that move the text in this direction. As is explained below, we also feel that if some of the advantages of small-group face-to-face teaching could be reflected in the written text, many students would find it beneficial.

One of the advantages of an oral approach to teaching, apparent to some extent in the lecture situation, and certainly in what are usually known as tutorials,[1] is the opportunity to follow the exposition of any particular point with an immediate short, but probing, question that helps to establish whether or not the student has grasped that point. This facility is not normally available when instruction is through a written medium, without having available at least the equipment necessary to access the contents of a storage disc.

In this book we have tried to go some way towards remedying this by making a non-standard use of footnotes. Some footnotes are used in traditional ways, to add a comment or a pertinent but not essential piece of additional information, to clarify a point by restating it in slightly different terms, or to make reference to another part of the text or an external source. However, about half of the nearly 300 footnotes in *this* book contain a question for the reader to answer or an instruction for them to follow; neither will call for a lengthy response, but in both cases an understanding of the associated material in the text will be required. This parallels the sort of follow-up a student might have to supply orally in a small-group tutorial, after a particular aspect of their written work has been discussed.

Naturally, students should attempt to respond to footnote questions using the skills and knowledge they have acquired, re-reading the relevant text if necessary, but if they are unsure of their answer, or wish to feel the satisfaction of having their correct response confirmed, they can consult the specimen answers given in Appendix H. Equally, footnotes in the form of observations will have served their purpose when students are consistently able to say to themselves "I didn't need that comment – I had already spotted and checked that particular point".

One further feature of the present volume is the inclusion at the end of each chapter, just before the problems begin, of a summary of the main results of that chapter. For some areas, this takes the form of a tabulation of the various case types that may arise in the context of the chapter; this should help the student to see the parallels between situations which in the main text are presented as a consecutive series of often quite lengthy pieces of mathematical development. It should be said that in such a summary it is not possible to state every detailed condition attached to each result, and the reader should consider

1 But in Cambridge are called "supervisions"!

the summaries as reminders and formulae providers, rather than as teaching text; that is the job of the main text and its footnotes.

Finally, we note, for the record, that the format and number of problems associated with the various remaining chapters have not been changed significantly, though problems based on excised topics have naturally been omitted. This means that hints or abbreviated solutions to all 200 odd-numbered problems appear in this text, and fully worked solutions of the same problems can be found in an accompanying volume, the *Student Solution Manual* for *Essential Mathematical Methods for the Physical Sciences*. Fully worked solutions to all problems, both odd- and even-numbered, are available to accredited instructors on the password-protected website www.cambridge.org/essential. Instructors wishing to have access to the website should contact solutions@cambridge.org for registration details.

Review of background topics

As explained in the Preface, this book is intended for those students who are pursuing a course in mathematical methods, but for whom it is not their first engagement with mathematics beyond high school level. Typically, such students will have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations. The emphasis in the text is very much on developing the *methods* required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields; the basic mathematical "tools" that the student is presumed to have mastered are therefore not discussed in any detail.

However this introductory note and the associated appendix (Appendix A) are included both to act as a reference (or reminder) and to be an indicator of any presumed, but missing, topics in the student's background knowledge. The appendix consists of summary pages for ten major topic areas, ranging from powers and logarithms at one extreme to first-order ordinary differential equations at the other. The style they adopt is identical to that used for the chapter summary pages in the 17 main chapters of the book. It should be noted that in such summaries it is not possible to state every detailed condition attached to each result. In the areas covered in Appendix A, there are very few subtle situations to consider, but the reader should be aware that they may exist.

Naturally, being only summaries, the various sections of the appendix will not be sufficient for the student who needs to catch up in some area, to learn the particular topics from scratch. A more elementary text will clearly be needed; *Foundation Mathematics for the Physical Sciences* written by the current authors would be one such possibility.

Contents

1 Matrices and vector spaces

In so far as vector algebra is concerned (see the summary in Section A.9 of Appendix A), a *vector* can be considered as a geometrical object which has both a magnitude and a direction, and may be thought of as an arrow fixed in our familiar three-dimensional space. This space, in turn, may be defined by reference to, say, the fixed stars. This geometrical definition of a vector is both useful and important since it is *independent* of any coordinate system with which we choose to label points in space.

In most specific applications, however, it is necessary at some stage to choose a coordinate system and to break down a vector into its *component vectors* in the directions of increasing coordinate values. Thus for a particular Cartesian coordinate system (for example) the component vectors of a vector $\mathbf{a}$ will be $a_x\mathbf{i}$, $a_y\mathbf{j}$ and $a_z\mathbf{k}$ and the complete vector will be

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{1.1}$$

Although for many purposes we need consider only real three-dimensional space, the notion of a vector may be extended to more abstract spaces, which in general can have an arbitrary number of dimensions N. We may still think of such a vector as an "arrow" in this abstract space, so that it is again *independent* of any (N-dimensional) coordinate system with which we choose to label the space. As an example of such a space, which, though abstract, has very practical applications, we may consider the description of a mechanical or electrical system. If the state of a system is uniquely specified by assigning values to a set of N variables, which could include angles or currents, for example, then that state can be represented by a vector in an N-dimensional space, the vector having those values as its components.[1]

In this chapter we first discuss general *vector spaces* and their properties. We then go on to consider the transformation of one vector into another by a linear operator. This leads naturally to the concept of a *matrix*, a two-dimensional array of numbers. The properties of matrices are then developed and we conclude with a discussion of how to use these properties to solve systems of linear equations and study some oscillatory systems.

1 This is an approach often used in control engineering.

1.1 Vector spaces

A set of objects (vectors) $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ is said to form a *linear vector space* V if:

(i) the set is closed under commutative and associative addition, so that

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}, \tag{1.2}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}); \tag{1.3}$$

(ii) the set is closed under multiplication by a scalar (any complex number) to form a new vector $\lambda\mathbf{a}$, the operation being both distributive and associative so that

$$\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}, \tag{1.4}$$

$$(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}, \tag{1.5}$$

$$\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}, \tag{1.6}$$

where λ and μ are arbitrary scalars;

(iii) there exists a *null vector* $\mathbf{0}$ such that $\mathbf{a}+\mathbf{0}=\mathbf{a}$ for all $\mathbf{a}$;

(iv) multiplication by unity leaves any vector unchanged, i.e. $1\times\mathbf{a}=\mathbf{a}$;

(v) all vectors have a corresponding *negative vector* $-\mathbf{a}$ such that $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$. It follows from (1.5) with $\lambda=1$ and $\mu=-1$ that $-\mathbf{a}$ is the same vector as $(-1)\times\mathbf{a}$.

We note that if we restrict all scalars to be real then we obtain a *real vector space* (an example of which is our familiar three-dimensional space); otherwise, in general, we obtain a *complex vector space*. We note that it is common to use the terms "vector space" and "space", instead of the more formal "linear vector space".

The *span* of a set of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{s}$ is defined as the set of all vectors that may be written as a linear sum of the original set, i.e. all vectors

$$\mathbf{x}=\alpha\mathbf{a}+\beta\mathbf{b}+\cdots+\sigma\mathbf{s} \tag{1.7}$$

that result from the infinite number of possible values of the (in general complex) scalars $\alpha, \beta, \ldots, \sigma$. If $\mathbf{x}$ in (1.7) is equal to $\mathbf{0}$ for some choice of $\alpha, \beta, \ldots, \sigma$ (not *all* zero), i.e. if

$$\alpha\mathbf{a}+\beta\mathbf{b}+\cdots+\sigma\mathbf{s}=\mathbf{0}, \tag{1.8}$$

then the set of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{s}$, is said to be *linearly dependent*. In such a set at least one vector is redundant, since it can be expressed as a linear sum of the others. If, however, (1.8) is not satisfied by *any* set of coefficients (other than the trivial case in which all the coefficients are zero) then the vectors are *linearly independent*, and no vector in the set can be expressed as a linear sum of the others.

If, in a given vector space, there exist sets of N linearly independent vectors, but no set of $N+1$ linearly independent vectors, then the vector space is said to be N-dimensional. In this chapter we will limit our discussion to vector spaces of finite dimensionality.