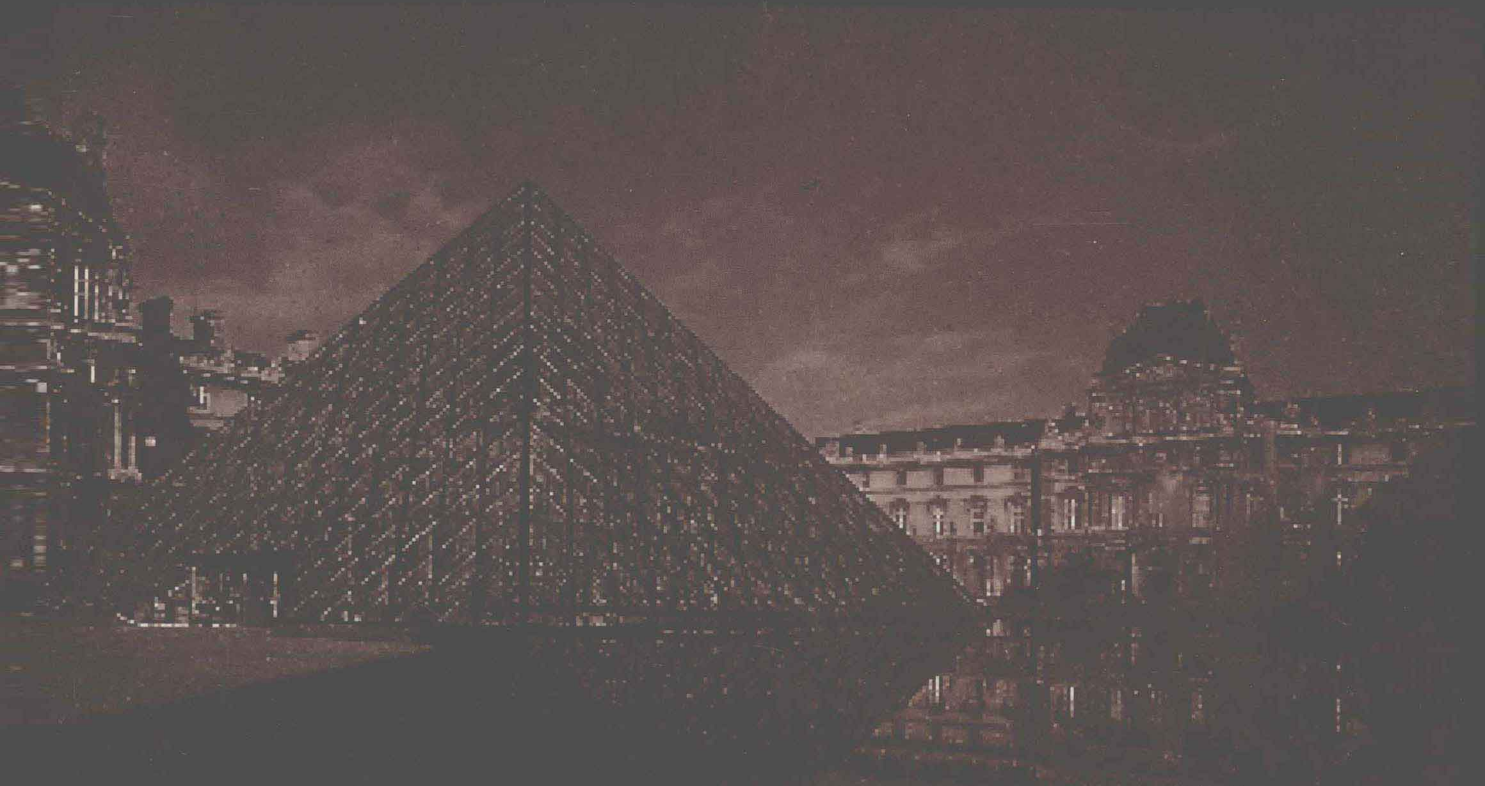
CALCULUS EXPLORATIONS WITH MAPLE

FRANK G. HAGIN & JACK K. COHEN



CALCULUS

ERALD L. BRADLEY / KARL J. SMITH

Calculus Explorations with *Maple*

Frank G. Hagin
Jack K. Cohen

to Accompany

CALCULUS

Gerald L. Bradley
Claremont McKenna College

Karl J. Smith
Santa Rosa Junior College

Prentice Hall
Englewood Cliffs, New Jersey 07632

Project Manager: ***Joanne E. Jimenez***
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A Simon & Schuster Company
Englewood Cliffs, New Jersey 07632

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-328576-6

PRENTICE-HALL INTERNATIONAL (UK) LIMITED, LONDON
PRENTICE-HALL OF AUSTRALIA PTY. LIMITED, SYDNEY
PRENTICE-HALL CANADA INC. TORONTO
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SIMON & SCHUSTER ASIA PTE. LTD., SINGAPORE
EDITORA PRENTICE-HALL DO BRASIL, LTDA., RIO DE JANEIRO

REMARKS TO STUDENTS

Maple is an important tool for working scientists. The exact such tool that you will use when you begin your career will no doubt be different, but it makes sense to begin developing skills in using symbolic mathematical computer programs and/or graphing calculators as soon as possible. However, remember that your primary task is to learn calculus, not to become a whiz at symbolic computation. Thus, the projects in this manual focus heavily on a small group of commands and only occasionally "open a door" to more advanced usage. Be aware that *Maple* affords a rich programming environment and that our usage is only a tiny bit of what is available.

We have tried to ease the burden of learning *Maple* by providing online Notebooks containing statements close to those that you will need to complete the project. The idea is to copy and paste these commands into your working Notebook. At the beginning, you will only have to do a small amount of editing on the copied commands to create the ones required by the project. As time goes on, we will shift a bit more of the burden onto you, so that you learn enough of the syntax to function on your own as you use *Maple* in other courses and for your own scientific explorations. If you find that your editing has created problems, it is often easier to recopy the command from the online Notebook than to figure out what *Maple* is complaining about.

However, there is no denying that a complex system like *Maple* will occasionally (we *hope* no more than occasionally!) frustrate you. Please try to keep a sense of humor about the human-computer interface.

This is not a textbook in *Maple*—we have told you only enough to get along. In fact, in some cases, we've written statements and not given an explanation. We assume that you are willing to try things out and observe what they do. For example, observe the difference in *Maple*'s response when we replace the semicolon by a colon in the statement `a := 2;`

```
a := 2;
     2

a := 2:
```

In the projects, we never stopped to explain that the colon prevented *Maple* from echoing the result. We assumed you would find out by trying it. (And now we *have* told you, after all.)

We hope that these projects will add to your calculus course—both by increasing your knowledge and by increasing your interest.

Most projects consist of three parts:

Before the Lab. Relevant reading and warm-up problems. These problems are important to the course as well as to the project.

In the Lab. Problems requiring *Maple*. Your instructor will tell you which problems to do. This section should take about an hour of lab work.

After the Lab. You'll be asked to reflect on what your computations revealed.

This Project Booklet includes:

1. An introductory tutorial on the basics of using *Maple* in a Notebook environment. Our version is explicitly written for the NeXT environment—other *Maple* Notebook environments are similar.

2. Weekly projects covering a three semester calculus course—there are "extra" projects to allow for differing taste and for variety of choice between semesters or sections.

3. Summaries of *Maple* statements.

4. A diskette containing a *Maple* "notebook" for each project containing *Maple* statements or code serving as copy and paste templates.

Your instructor may well shorten some of the projects by telling you to omit certain exercises. We suggest that you read over the omitted exercises and give brief thought as to how you would tackle them. You will also note that we have labeled certain projects as "optional." This designation is given to projects that either use advanced *Maple* commands that won't be needed in the sequel, or that involve a greater degree of exploration on your part. Again, even if your instructor omits some or all of these projects because of time pressure, we suggest that you read them—you may well want to refer back to these projects later in your academic or scientific career.

Your Authors welcome your comments for improving this manual. Our Internet addresses are:
jkc@keller.mines.colorado.edu
fhagin@newton.mines.colorado.edu

REMARKS TO INSTRUCTORS

For the past three years at the Colorado School of Mines, all 500-plus freshmen have taken a calculus course using major portions of the material you have in your hands. Your authors each have over 25 years of teaching experience at all university levels from freshman to Ph. D. level graduate courses. We have participated fully in all aspects of university life and are well published applied mathematicians. However, our recent experience with this calculus program ranks among the most exciting and rewarding of our careers—it added a distinct element of fun compared with our previous calculus experiences. However, while "fun" is not to be underestimated as a motivating force, we have also kept firmly in mind a number of more concrete pedagogical objectives. We explain our viewpoint in the following paragraphs.

Our guiding star is that we are teaching calculus, not *Maple*. This has important implications. For example, to our mind, premature reliance on constructs such as `limit` and `fsolve` can easily degenerate the *Maple* experience to being a high tech analog of looking up the answer in the back of the book. These "black boxes" don't help teach calculus, so we relegate them to a secondary role. On the other hand, we believe that students should be aware of these tools for use *after* they master the concepts of calculus, so we do include some projects devoted to such commands. The instructor notes to such projects will clearly state these projects are not of primary importance to learning calculus and may be skipped. In particular, succeeding projects will *not* depend on covering these "black box" projects. Consider at least making some reading assignments in some of these projects, but don't let them deflect you from your main task of teaching calculus.

To minimize the numerous syntax errors that occur when the neophyte tries to type in unfamiliar statements, we provide on-line template commands that allow the student to copy, paste and then do only minor editing to obtain the needed adaptation of the command. While we do not emphasize *Maple* syntax, over time we gradually make the students sensitive to it by gradually decreasing the amount of template help for the core *Maple* commands that have already occurred several times earlier. Occasionally, we use *Maple* commands outside the minimal core to "open doors" for students—but these more advanced commands are always supported fully by a template.

We have firm objectives in teaching calculus and these projects are designed to reinforce them:

Solve general problems. Instead of solving problems with numbers that make the answers "come out nice," solve families of problems (i.e. problems with parameters).

Emphasize fundamentals. Don't rush through hundreds of topics, instead take time with the topics that deserve it, and train students to *use* their text, when necessary, to look up topics that are less important.

Empower students. The picture of two or three students huddled around a computer (cooperative education) is often a better picture than 35 students busily taking notes in a lecture hall. We allow our students to hand in joint work for all the *Maple* projects—and we encourage you to think about doing this too.

As stated above, most projects consist of three parts: **Before the Lab**, **In the Lab** and **After the Lab**. The intent of this division is to make the Laboratory experience meaningful.

For each project, the instructor's Answer Booklet contains:

a) Answers (naturally!).

b) An introductory statement of the goals of the project.

c) Suggestions for ways to shorten the project (you'll probably want to vary the assignments from semester to semester or section to section).

d) Occasional Instructor Notes about likely student questions, etc.

The Answer Booklet is accompanied by an instructor diskette containing the commands needed to reproduce the results cited in the answers and a set of *Maple* utilities for constructing classroom demonstrations.

We have been generous in both the number of projects and their contents. For most projects, we list exercises that can be omitted to yield a shorter project. We have also labeled certain projects as "optional." This designation is given to projects that either use "black box" *Maple* commands that won't be needed in the sequel or that introduce modest "discovery" explorations. These projects can be among the most exciting, so we hope you won't omit *all* the optional projects. Similarly, we do not expect that you will assign *all* the regular projects. Also, notice that the generosity in projects is greater earlier in the Manual—this is to accommodate courses with differing priorities. Consequently, you may wish to consider assigning an "optional" project that appears early in the Manual in (say) the third semester, with a view towards increasing competence in *Maple* at that later time.

ACKNOWLEDGMENTS

Over these past three years, we have written numerous worksheets and projects containing problems to be solved using *Maple*. Most of them have significant original content, but many have been motivated by or use material from articles written in teaching-oriented journals. In particular, many of our project questions originated from routine textbook exercises. Similar exercises could be found in many calculus texts, but since we used the popular Edwards and Penney Third Edition during the past three years, often the original routine kernel exercise was drawn from this excellent text and we wish to acknowledge that.

We owe a great debt to our colleagues on the "Calculus Team" at the Colorado School of Mines, who willingly tested earlier versions of these projects. In particular, the advice and encouragement of the team coordinator, Barbara Bath, was invaluable. We also wish to acknowledge our grader, Shelby Worley, for his astute comments about student difficulties with the earlier versions of these projects. Our questions are distinctly more "student friendly" because of his efforts. Finally, we give our thanks to the students of the School of Mines for their enthusiasm and advice.

ADDRESSES

We welcome your comments for improving this manual.

Jack K. Cohen
Center for Wave Phenomena
Colorado School of Mines
Golden, CO 80401
jkc@keller.mines.colorado.edu

Frank G. Hagin
Department of Mathematics
Colorado School of Mines
Golden, CO 80401
fhagin@newton.mines.colorado.edu

Contents

BEFORE THE LAB

Be sure to carefully read the online file "Tutorial.ms" before attempting this project. **Tip**: The on-line notebooks that accompany these projects have "templates" for you to "copy and paste" from. This avoids little errors like using round brackets when you should use square brackets. Eventually, you'll have to write expressions on your own, but, for now, we'll give lots of help to minimize the inevitable frustration of learning a new language (*Maple*).

IN THE LAB

Exercise 1 Use *Maple* to evaluate $\sin(\pi/3)$ *numerically*. Verify (somehow) that the answer is correct.

Exercise 2 Evaluate $\sqrt{2}$ numerically to 16 "significant figures" (i.e, use 16 as the optional second argument in the numerical command, `evalf`).

Exercise 3 Define the functions $f(x) = x^2 + 1$ and $g(x) = x^3 - x^2 - 9x + 9$ and plot them on the same graph. From your graph estimate the three values of x where the plots cross.

Exercise 4 Form the "rational" function, $r(x) = g(x)/f(x)$ and plot it for $-20 \le x \le 20$. Make a hand sketch of what you see.

Exercise 5 Define the function $G(x) = x^3 + 2x^2 - 15x - 30$ and plot G on several x-intervals to get a good idea of its behavior.

a) Zoom in on the largest x-value for which $G = 0$. Does the function become "almost linear" as you zoom in?

b) Use the `factor` command to find the exact x-values at which $G = 0$.

c) Define a new rational function, $R = G(x)/F(x)$, where $F(x) = x^4 + 3$. Decide if R is "almost linear" for x large (as was the case with the r discussed earlier).

AFTER THE LAB

Exercise 6 Explain why the graphs of $g(x)$ and $r(x)$ are similar for "small" x.

Exercise 7 The graph of $r(x)$ looks linear for "large" x. Can you figure out *what* linear function approximates r for large x?

BEFORE THE LAB

In high school, you learned the solution formula for the quadratic equation, $Ax^2 + Bx + C = 0$. However, you may not know any methods for solving higher degree or transcendental equations (i.e., equations involving trig functions, logs, etc.). Oftentimes, such equations can be approximately solved by using `plot` to "zoom in" on the desired root as you will do in the laboratory portion of this Project.

Exercise 1 In each case, write an equation of the straight line L that is described:

a) L passes through $(1, 5)$ and is parallel to the line with equation $2x + y = 10$.

b) L passes through $(-2, 4)$ and is perpendicular to the line with equation $x + 2y = 17$.

Remark: You may find it useful to check your text for the various forms used to describe straight lines: slope-intercept, point-slope, and two-point, to find the easiest one to use for the above problems.

Exercise 2 You could make up a million problems like the ones in the previous exercise. It makes more scientific sense to solve *general* problems (though "warm-ups" with numerical values are often useful). Solve the following generalized version of the first part of Exercise 1:

> Write the equation of the straight line L that passes through the point (p, q) and is parallel to the line with equation $ax + by = c$. Check that your answer "works" with the specific numbers in part (a) of Exercise 1.

Exercise 3 State the general problem corresponding to the second part of Exercise 1, solve it, and check with the specific example.

We will soon learn that:

The slope of the tangent line to the parabola $y = x^2$ at $x = a$ is $2a$.

Thus, the slope of this parabola at $x = 1$ is 2, the slope at $x = 3$ is 6 and so on. Please accept this as a fact for now.

Illustrative Exercise: Find the equation of the line through the point $P(2, 0)$ that is normal to the parabola $y = x^2$.

> **Solution**: Draw a figure (see Figure 1) that includes a rough sketch of the normal from the given point. Introduce the letter a to denote the x-coordinate of the point where the normal cuts the parabola. Thus, the point (a, a^2) is on both the parabola and the normal. The slope of the parabola at $x = a$ is $2a$, so the slope of the normal is the negative reciprocal value $-1/2a$. Equating this latter slope to the slope from (a, a^2) to the given point $(2, 0)$ yields

$$-\frac{1}{2a} = \frac{a^2 - 0}{a - 2}$$

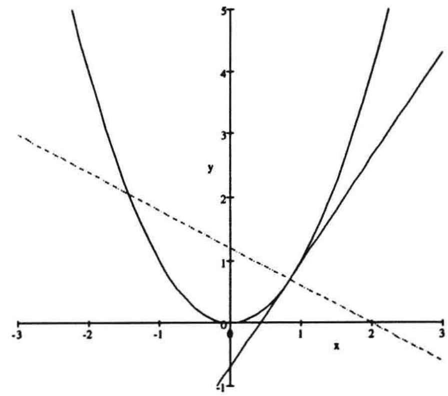

FIG. 1. Sketch for the Illustrative Exercise.

leading to the *cubic* equation,

$$2a^3 + a - 2 = 0$$

for a. We can see from our sketch that $a \approx 1$, but this is *not* the exact root (try it). Instead of resorting to trial and error, we can **plot** the function near $a = 1$ and then "zoom in" on the root. The three successive **plot** commands

```
plot(2*a^3 + a - 2, a = 0.7..1.1);
plot(2*a^3 + a - 2, a = 0.82..0.85);
plot(2*a^3 + a - 2, a = 0.834..0.836);
```

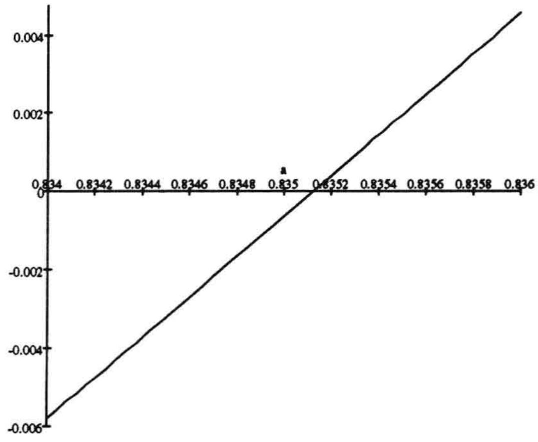

FIG. 2. Zooming in on the solution of the Illustrative Exercise.

produce the results shown respectively from top to bottom in Figure 2. From the bottom graph, we see that $a \approx .835$. Now is the time to be careful—we've worked hard to get a good approximation to a, but that is *not* quite what the exercise called for! The correct solution to the problem is to produce the requested *normal* from $(2, 0)$ to the parabola. This line is given by $\boxed{y = -\frac{1}{2a}(x-2), \quad \text{where } a \approx .835}$.

Exercise 4 One problem solving maxim we've ignored in the above solution is that of trying to avoid the use of specific numbers. To maintain generality, let (p, q) denote the given point (i.e replace $(2, 0)$ by (p, q)). Show that the cubic equation giving the point $x = a$ where the normal from (p, q) intersects the parabola $y = x^2$ is $\boxed{2a^3 - (2q-1)a - p = 0}$. It's good practice to check that this agrees with the specific case worked out in the Illustrative Exercise, so do it!

IN THE LAB

Exercise 5 In reference to the previous exercises, find the normal when $(p, q) = (4, 0)$. If you are uncertain how to proceed or how to present your solution, re-read the Illustrative Exercise.

Exercise 6 In reference to the previous problems, when $(p, q) = (1, 17/4)$ show that there are *three* normals and determine them by doing three "zooms." Notice that the given point is *inside* the parabola.

Notice how much more convenient and less error prone it is to use the *general* form for the cubic instead of setting up and simplifying the fractions for each specific point $(4, 0)$, $(1, 17/4)$, ...

Exercise 7 Show that the cubic equation in Exercise 6 has the *exact* solution $a = 2$. Use `Factor` on your cubic to obtain a quadratic equation. Solve the quadratic to get all three roots of the cubic exactly. Check that the exact solutions agree with the ones you found by the zoom method.

AFTER THE LAB

Exercise 8 In light of your experience with the previous exercises, comment on the efficiency of the zoom method if highly accurate solutions (say 10 decimal place accuracy) are required.

Exercise 9 Comment on the efficiency of the zoom method if solutions are required for *many* different points (p, q).

Exercise 10 In light of the results obtained in the lab exercises, a natural conjecture is that there are three normals for points interior to the parabola $y = x^2$ and only one for exterior points. Investigate the truth of this conjecture for the special case when the point is on the y-axis (that is, $p = 0$).

Exercise 11 As a review of hand methods, repeat Exercise 7 using the algebraic analog of long division to divide the known factor out of the cubic and thus again obtain the quadratic equation for the remaining roots.

BEFORE THE LAB

Exercise 1 You land in your space ship on a spherical asteroid. Your partner walks 400 meters away along the smooth surface carrying a one meter rod, and thereby vanishes over the horizon. When she places one end on the ground and holds the rod straight up and down, you, lying on your stomach, can just barely see the tip of the rod. The ultimate quest is to determine the radius r of the asteroid, but, for now, just draw a careful diagram and label it with the quantities you think will be useful in getting an equation involving only r and known quantities.

Exercise 2 After checking with your instructor that your diagram for the asteroid problem is correct, use it to derive the equation for the radius r in terms of the given quantities. Again use symbols instead of specific numbers and check your equation with your instructor before going on with the project.

IN THE LAB

Building a table in *Maple*

There are several ways to build a table. Here we use the **for** command:

```
   for x from 0 by 0.2 to 1.0 do
      lprint(x, cos(x) -x)
   od;
 0    1
.2   .7800665778
.4   .5210609940
.6   .2253356149
.8   -.1032932907
1.0   -.4596976941
```

Recall one could also use, in the above **for** statement, the set notation {x, cos(x)-x}, or print(x, cos(x)-x).

Exercise 3 Give an intuitive explanation of why there is a solution of $\cos x = x$ in the interval $0.6 \leq x \leq 0.8$; and use a table to determine this solution with error less than 0.05.

The Asteroid Problem—continued

Exercise 4 You now know the correct equation for r. Using `plot` and/or tables, find the solution to one decimal place accuracy for the given numerical values.

Comment: How much would you contribute to the Indigent Math Professors' Support Fund for an easily derived estimate of the solution to within 1 meter? Actually, this fee is covered in your tuition for this course—stay tuned. Without questioning that `plot` and tabling are terrific tools, your experience with the current problem may cause you to wonder if there aren't some better methods of solving equations. If so, you will be glad to hear that you will, indeed, be learning about more advanced solution methods this semester.

AFTER THE LAB

We will be returning to the asteroid problem several times later in the semester, so it is worthwhile to reflect on your experience so far:

Exercise 5 In light of your Laboratory investigations, discuss how you could have derived the answer more efficiently.

Exercise 6 In light of your Laboratory investigations, discuss the virtues and limitations of the `plot` command and tabling in solving equations.

BEFORE THE LAB

Your text shows that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. This is a special case of the **Power Law**:

$$f(x) = x^n \Longrightarrow f'(x) = nx^{n-1} \qquad \text{for } n = 1, 2, \ldots.$$

The `for` loop

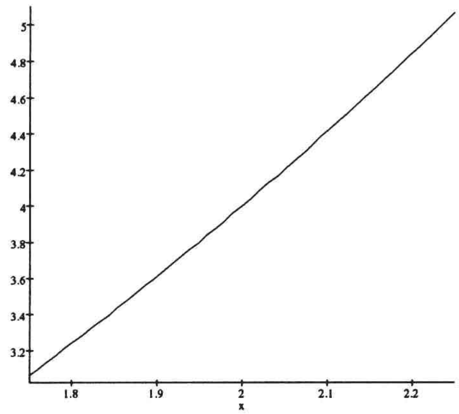

FIG. 3. First four zooms of x^2 at $x = 2$.

It is tiring to generate a series of "zoom" **plots** one at a time. Here is a **for** loop that zooms in on $x = 2$ for $f(x) = x^2$:

```
a  := 2.0: h := 4.0:
for k to 5 do
   h := h/2:
   plot(x^2, x = a-h..a+h)
od;
```

On each pass through the loop, a **plot** is generated with a plot range varying from $x = a - h$ to $x = a + h$. The quantity h is set to 4 before the loop. As we enter the loop, it is cut in half, so the first plot goes from $x = a - 2$ to $x = a + 2$. Before the second plot, h is cut in half again, so that the second plot goes from $x = a - 1$ to $x = a + 1$. Similarly, the third plot goes from $x = a - 1/2$ to $x = a + 1/2$ and so on. Since a is set to 2 before the loop, the actual successive numerical plot ranges are $[0, 4]$, $[1, 3]$, $[1.5, 2.5]$, ... So we are, indeed, "zooming" in on $x = 2$.

The above `for` loop is more concise than the usual "`for k from 1 by 1 to 5 do ...`"; since we are starting with $k = 1$ and taking steps of 1, the 'from' and 'by' parts can be omitted. (That is, 1 is the default value for the 'from' and 'by' clauses). Since we are to pass through the loop 5 times the final plot range is $[1.875, 2.125]$. The **plots** from the first four zooms are shown in Figure 3 and the final zoom is shown in Figure 4.

Tip: It is sensible to set the "iterator" limit (here, 5) to a small number (like 2) until you are *sure* the loop is doing what you want.

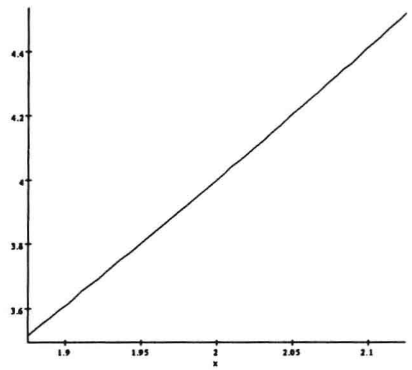

FIG. 4. Final zoom of x^2 at $x = 2$.

Exercise 1 Use the output of the final step in the `for` loop above shown in Figure 4 to verify that the slope of $f(x) = x^2$ at $x = 2$ is what it should be. Watch out for the fact that *Maple* rescales the `plot`—you should use the *numbers* on the axes for your computations, *not* the apparent slope. Reproduce the figure as a rough sketch and draw any necessary lines on your sketch. Make the best estimate you can of the slope and also state the error in your approximation to the exact slope. **Warning**: Take into account the fact that *Maple* does not always put the origin where the axes cross—that is, be sure to compute the difference quotient, not just y/x.

IN THE LAB

Exercise 2 Use *Maple* to verify the power law for $n = 3$ and $n = 4$. Method: Define a suitable `f(x)` and simulate the standard simplification of the difference quotient by using the command `expand((f(x + h) - f(x))/h)`. Then evaluate the resulting simple limit as $h \to 0$ any way you like.

Exercise 3 In this exercise, we find the tangent line to a curve using the idea that the tangent line is the *limit* of secant lines. As a nontrivial example, we use the function:

$$f(x) = \frac{(x^3 - 5)(x^2 - 1)}{x^2 + 1}$$

and find the tangent line at the specific point $(2, f(2)) = (2, 1.8)$.

To this end, consider two points on the graph, namely $(2, f(2))$ and $(b, f(b))$, where b is close to, but not equal to 2. The slope of the "secant" line (the line that connects the two points) is $(f(b) - f(2))/(b - 2)$. Let's start with $b = 2.5$. To calculate the secant line, you execute the following commands:

```
f := x -> (x^3 - 5)*(x^2 - 1) / (x^2 + 1);
b := 2.1:
slope := (f(b) - f(2)) / (b - 2.0):
secant := f(2.0) + slope*(x - 2.0);
```