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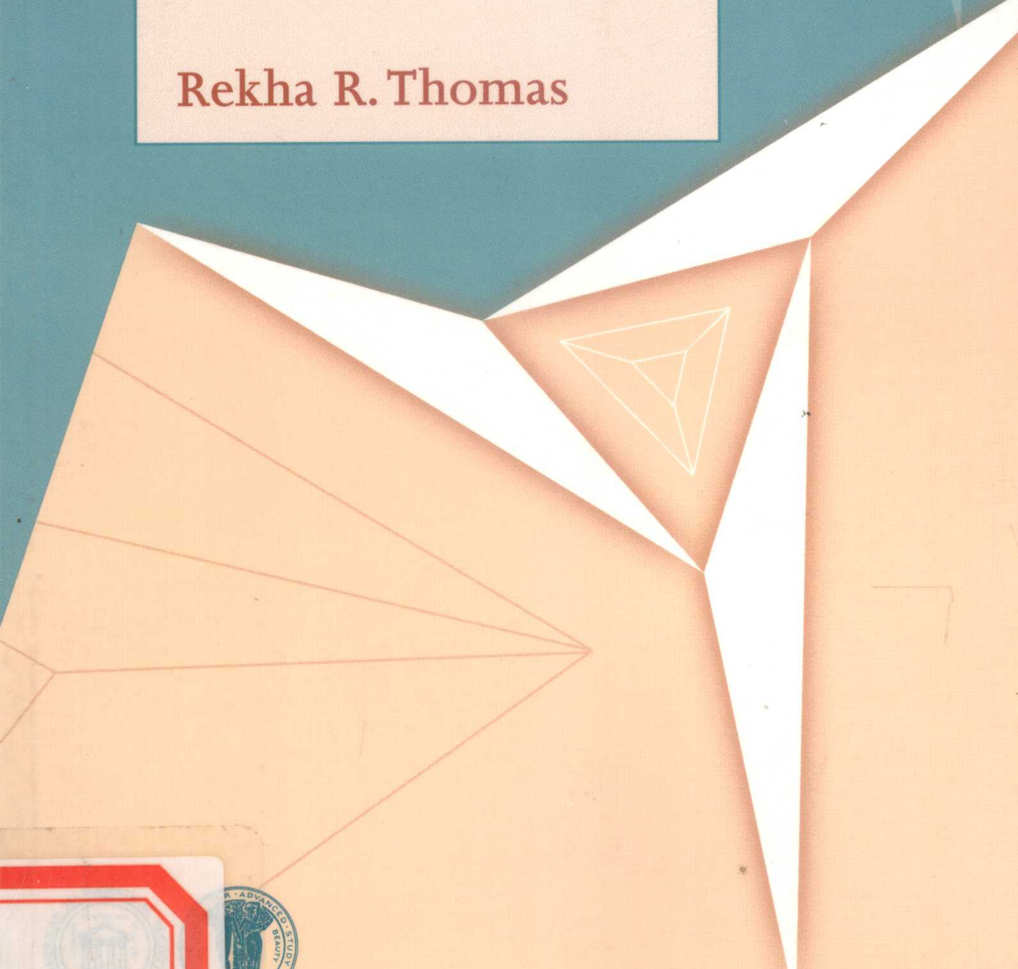
Lectures in Geometric Combinatorics

Rekha R. Thomas



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IAS/Park City Mathematics Institute

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The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, mathematics education researchers, and undergraduate faculty. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the High School Teacher Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Program deal with this topic as well. Lecture notes from the Graduate Summer School are published each year in the IAS/Park City Mathematics Series. Course materials from the Undergraduate Program, such as the current volume, are now being published as part of the IAS/Park City Mathematical Subseries in the Student Mathematical Library. We are happy to make available more of the excellent resources which have been developed as part of the PCMI.

John Polking, Series Editor
February 20, 2006

Preface

These lectures were prepared for the advanced undergraduate course in *Geometric Combinatorics* at the Park City Mathematics Institute in July 2004. Many thanks to the organizers of the undergraduate program, Bill Barker and Roger Howe, for inviting me to teach this course. I also wish to thank Ezra Miller, Vic Reiner and Bernd Sturmfels, who coordinated the graduate research program at PCMI, for their support. Edwin O'Shea conducted all the tutorials at the course and wrote several of the exercises seen in these lectures. Edwin was a huge help in the preparation of these lectures from beginning to end.

The main goal of these lectures was to develop the theory of convex polytopes from a geometric viewpoint to lead up to recent developments centered around secondary and state polytopes arising from point configurations. The geometric viewpoint naturally relies on linear optimization over polytopes. Chapters 2 and 3 develop the basics of polytope theory. In Chapters 4 and 5 we see the tools of Schlegel and Gale diagrams for visualizing polytopes and understanding their facial structure. Gale diagrams have been used to unearth several bizarre phenomena in polytopes, such as the existence of polytopes whose vertices cannot have rational coordinates and others whose facets cannot be prescribed. These examples are described in Chapter 6. In Chapters 7–9 we construct the secondary polytope of a graded point configuration. The faces of this polytope index the regular subdivisions of the configuration. Secondary polytopes appeared in the literature in the early 1990's and play a crucial role in combinatorics, discrete optimization and algebraic geometry. The secondary polytope of a point configuration is naturally refined by the state polytope of the toric ideal of the configuration. In Chapters 10–14 we establish this relationship. The state polytope of a toric ideal arises from the theory of Gröbner bases, which is developed in Chapters 10–12. Chapter 13 establishes the connection between the Gröbner bases of a toric ideal and the regular triangulations of the point configuration defining the ideal. Finally, in Chapter 14 we construct the state polytope of a toric ideal and relate it to the corresponding secondary polytope.

These lectures are meant to be self-contained and do not require any background beyond basic linear algebra. The concepts needed from abstract algebra are developed in Chapters 1, 10, 11 and 12.

I wish to thank Tristram Bogart, Ezra Miller, Edwin O'Shea and Alex Papazoglu for carefully proofreading many parts of the original manuscript. Ezra made several important remarks and corrections that have greatly benefited this final version. Many thanks also to Sergei Gelfand and Ed Dunne at the AMS office for their patience and help in publishing this book. Lastly, I wish to thank Peter Blossey for twenty-four hour technical assistance in preparing this book.

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Rekha R. Thomas
Seattle, January 2006

Contents

Chapter 1

Abstract Algebra: Groups, Rings and Fields

This course will aim at understanding *convex polytopes*, which are fundamental geometric objects in combinatorics, using techniques from algebra and discrete geometry. Polytopes arise everywhere in the real world and in mathematics. The most famous examples are the Platonic solids in three-dimensional space: *cube*, *tetrahedron*, *octahedron*, *icosahedron* and *dodecahedron*, which were known to the ancient Greeks. The natural first approach to understanding polytopes should be through geometry as they are first and foremost geometric objects. However, any experience with visualizing geometric objects will tell you soon that geometry is already quite hard in three-dimensional space, and if one has to study objects in four- or higher-dimensional space, then it is essentially hopeless to rely only on our geometric and drawing skills. This frustration led mathematicians to the discovery that algebra can be used to encode geometry and, since algebra does not suffer from the same limitations as geometry in dealing with higher dimensions, it can serve very well as the language of geometry. A simple example of this translation can be seen by noting that, while it is hard to visualize vectors in four-dimensional space, linear algebra allows us to work with their algebraic incarnations $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$ and to manipulate them to find new quantities, such as the sum vector $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4) \in \mathbb{R}^4$ or the length of their difference vector $\sqrt{(v_1 - w_1)^2 + \cdots + (v_4 - w_4)^2}$. We use $\mathbb{R}$ for the set of real numbers.

These lectures will focus on techniques from linear and abstract algebra to understand the geometry and combinatorics of polytopes. We begin with some basic abstract algebra. The algebraically sophisticated reader should skip ahead to the next chapter and refer back to this chapter only as needed. The material in this lecture is taken largely from the book [**DF91**].

In linear algebra one learns about *vector spaces* over *fields*. Both of these objects are examples of a more basic object known as a *group*.

Definition 1.1. A set G along with an operation $*$ on pairs of elements of G is called a **group** if the pair $(G, *)$ satisfies the following properties:

(1) $*$ *is a binary operation on* G: This means that for any two elements $g_1, g_2 \in G$, $g_1 * g_2 \in G$. In other words, G is *closed* under the operation $*$ on its elements.

(2) $*$ *is associative*: For any three elements $g_1, g_2, g_3 \in G$, $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

(3) G *has an identity element with respect to* $*$: This means that there is an element $e \in G$ such that for all $g \in G$, $e * g = g * e = g$. If $*$ is addition, then e is usually written as 0. If $*$ is multiplication, then e is usually written as 1.

(4) *Every* $g \in G$ *has an inverse*: For each $g \in G$ there is an element $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. If $*$ is addition, then it is usual to write g^{-1} as $-g$.

It can be proved that the identity element in G is unique and that every element in G has a unique inverse. Let $\mathbb{Z}$ be the set of integers and let $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. The **multiplication table** of a finite group is a $|G| \times |G|$ array whose rows and columns are indexed by the elements of G and the entry in the box with row index g and column index g' is the product $g * g'$.

Exercise 1.2. Check that the following are groups. In each case, write down how the binary operation works, the identity element of the group, and the inverse of an arbitrary element in the group.

(1) $(\mathbb{Z}^n, +)$
(2) $(\mathbb{R}^*, \times)$
(3) $((\mathbb{R}^*)^n, \times)$

The above groups are all infinite. We now study two important families of finite groups that are useful in the study of polytopes.

The symmetric group S_n:
Recall that a **permutation** of n letters $1, 2, \ldots, n$ is any arrangement of the n letters or, more formally, a one-to-one onto function from the set $[n] := \{1, 2, \ldots, n\}$ to itself. Permutations are denoted by the small Greek letters σ, τ, etc. and they can be written in many ways. For instance, the permutation

$$\sigma : \{1,2,3\} \to \{1,2,3\} \; : \; 1 \mapsto 2, \; 2 \mapsto 1, \; 3 \mapsto 3$$

is denoted as either $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ or, more compactly, by recording just the last row as 213. Since permutations are functions, two permutations can be composed in the usual way that functions are composed: $f \circ g$ is the function obtained by first applying g and then applying f. The symbol $\circ$ denotes composition. Check that $213 \circ 321 = 312$, which is again a permutation. Let S_n denote the set of all permutations on n letters. Then $(S_n, \circ)$ is a group with $n!$ elements. We sometimes say that 312 is the *product* $213 \circ 321$.

Exercise 1.3. (1) Check that $(S_n, \circ)$ is a group for any positive integer n. What is the identity element of this group, and what is the inverse of a permutation $\sigma \in S_n$?

(2) List the elements of S_2 and S_3, and compute their multiplication tables.

Definition 1.4. The group $(G, *)$ is **abelian** if for all $g, g' \in G$, $g * g' = g' * g$.

Check that S_3 is not an abelian group. Do you see how to use this to prove that $(S_n, \circ)$ is not abelian for all $n \geq 3$?

We now study a second family of non-abelian groups. The **regular n-gon**, which is a polygon with n sides of equal length, is an example of a *polytope* in $\mathbb{R}^2$. *Regular* polygons have all sides of equal length and the same angle between any two adjacent sides. For instance, an equilateral triangle is a regular 3-gon, a square is a regular 4-gon, a pentagon with equal sides and angles is a regular 5-gon, etc.

The dihedral group D_{2n}:
The group D_{2n} is the *group of symmetries* of a regular n-gon. A **symmetry** of a regular n-gon is any rigid motion obtained by taking a copy of the n-gon, moving this copy in any fashion in three-dimensional space and placing it back down so that the copy exactly covers the original n-gon. Mathematically, we can describe a symmetry s by a permutation in S_n. Fix a cyclic labeling of the corners (*vertices*) of the n-gon by the letters $1, 2, \ldots, n$. If s puts vertex i in the place where vertex j was originally, then the permutation s sends i to j. Note that since our labeling was cyclic, s is completely specified by noting where the vertices 1 and 2 are sent. In particular, this implies that s cannot be any permutation in S_n.

How many symmetries are there for a regular n-gon? Given a vertex i, there is a symmetry that sends vertex 1 to i. Then vertex 2 has to go to either vertex $i - 1$ or vertex $i + 1$. Note that we have to add modulo n and hence $n + 1$ is 1 and $1 - 1$ is n. By following the first symmetry by a reflection of the n-gon about the line joining the center of the n-gon to vertex i, we see that there are symmetries that send 2 to either $i - 1$ or $i + 1$. Thus there are $2n$ positions that the ordered pair of vertices 1 and 2 may be sent to by symmetries. However, since every symmetry is completely determined by what happens to 1 and 2, we conclude that there are exactly $2n$ symmetries of the regular n-gon. These $2n$ symmetries are the n *rotations* about the center through $\frac{2\pi i}{n}$ radians for $1 \leq i \leq n$ and the n *reflections* through the n lines of symmetry. If n is odd, each symmetry line passes through a vertex and the midpoint of the opposite side. If n is even, there are $n/2$ lines of symmetry which pass through two opposite vertices and $n/2$ which perpendicularly bisect two opposite sides. The dihedral group D_{2n} is the set of all symmetries of a regular n-gon with the binary operation of composition of symmetries (which are permutations).

Example 1.5. Let $\square$ be a square with vertices $1, 2, 3, 4$ that are labeled counterclockwise from the bottom left vertex and centered about the origin in $\mathbb{R}^2$. Then its group of symmetries is the dihedral group

$$D_8 = \left\{ \begin{array}{ll} e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \\ r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, & r^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \\ r^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, & r \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \\ r^2 \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, & r^3 \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \end{array} \right\}$$

where r denotes counterclockwise rotation by 90 degrees about the origin and s denotes reflection about the horizontal axis.

Exercise 1.6. Fix a labeling of a regular n-gon (say counterclockwise, starting at some vertex). Let r denote counterclockwise rotation through $\frac{2\pi}{n}$ radians and let s denote reflection about the line of symmetry through the center of the n-gon and vertex 1. Then show the following.

(1) $D_{2n} = \{e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$.

(2) What are the inverses in the above group?

(**Hint**: (i) $1, r, r^2, \ldots, r^{n-1}$ are all distinct, (ii) $r^n = e$, (iii) $s^2 = e$, (iv) $s \neq r^i$ for any i, (v) $sr^i \neq sr^j$ for all $0 \leq i, j \leq n-1$, $i \neq j$, (vi) $sr = r^{-1}s$, (vii) $sr^i = r^{-i}s$, for $0 \leq i \leq n$.)

Exercise 1.7. Let G be the symmetries of a regular cube in $\mathbb{R}^3$. Show that $|G| = 24$.

Definition 1.8. A set R with two binary operations $+$ and $\times$ is called a **ring** if the following conditions are satisfied:

(1) $(R, +)$ is an abelian group,

(2) $\times$ is associative : $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in R$,

(3) $\times$ distributes over $+$: for all $a, b, c \in R$,

$$(a + b) \times c = (a \times c) + (b \times c), \text{ and } a \times (b + c) = (a \times b) + (a \times c).$$

If, in addition, R has an identity element with respect to $\times$, we say that R is a ring with identity. If $\times$ is commutative in R, then we say that R is a commutative ring. The identity of $(R, +)$ is the additive identity in R denoted as 0 while the multiplicative identity, if it exists, is denoted as 1. We will only consider commutative rings with identity.

Exercise 1.9.
(1) Show that $(\mathbb{Z}, +, \times)$ is a commutative ring with identity.
(2) Let M_n denote the set of $n \times n$ matrices with entries in $\mathbb{R}$. Then show that under the usual operations of matrix addition and multiplication, M_n is a non-commutative ring with identity. Is $(M_n, \times)$ a group?

Definition 1.10. A **field** is a set F with two binary operations $+$ and $\times$ such that both $(F, +)$ and $(F^* := F \backslash \{0\}, \times)$ are abelian groups and the following distributive law holds:

$$a \times (b + c) = (a \times b) + (a \times c), \text{ for all } a, b, c \in F.$$

Let $\mathbb{C}$ denote the set of complex numbers and let $\mathbb{Q}$ denote the set of rational numbers.

Exercise 1.11. Check that $(\mathbb{C}, +, \times)$, $(\mathbb{R}, +, \times)$, $(\mathbb{Q}, +, \times)$ are fields while $(\mathbb{Z}, +, \times)$ and $(M_n, +, \times)$ are not fields.

Where does a vector space fit in the above hierarchy?

Chapter 2

Convex Polytopes: Definitions and Examples

In this chapter we define the notion of a convex polytope. There are several excellent books on polytopes. Much of the material on polytopes in this book is taken from [**Grü03**] and [**Zie95**]. We start with an example of a family of convex polytopes.

Example 2.1. Cubes: The following is an example of the familiar three-dimensional cube:

$$C_3 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ 0 \le x_3 \le 1 \end{array} \right\}.$$

The cube C_3 has volume one and edges of length one. By translating this cube around in $\mathbb{R}^3$, we see that there are infinitely many three-dimensional cubes (3-cubes) of volume one and edges of length one in $\mathbb{R}^3$. If you are interested in studying the properties of these cubes, you might be willing to believe that it suffices to examine one member in this infinite family. Thus we pick the above member of the family and call it *the* three-dimensional **unit cube**.

The unit 3-cube is of course the older sibling of a square in $\mathbb{R}^2$. Again, picking a representative, we have *the* unit square (or the unit

2-cube):

$$C_2 := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{array} \right\}.$$

Going down in the family, we could ask who the 1-cube is. If we simply mimic the pattern, we might conclude that the unit 1-cube is the line segment:

$$C_1 := \{ \ (x_1) \in \mathbb{R} : \quad 0 \le x_1 \le 1 \ \}.$$

The baby of the family is the 0-cube $C_0 = \{0\} = \mathbb{R}^0$.

How about going up in the family? What might be the unit 4-cube? Again, simply mimicking the pattern, we might define it to be

$$C_4 := \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ 0 \le x_3 \le 1 \\ 0 \le x_4 \le 1 \end{array} \right\}.$$

Of course this is hard to visualize. In Chapter 4 we will learn about Schlegel diagrams that can be used to see C_4. Making life even harder, we could define the unit d-cube (the unit cube of dimension d) to be

$$C_d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ \vdots \\ 0 \le x_d \le 1 \end{array} \right\},$$

thus creating an infinite family of unit cubes $\{C_d : d \in \mathbb{N}\}$. The symbol $\mathbb{N}$ denotes the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$. Every member of this family is a convex polytope.

Definition 2.2. A set $C \subseteq \mathbb{R}^d$ is **convex** if for any two points $\mathbf{p}$ and $\mathbf{q}$ in C, the entire line segment joining them, $\{\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} : 0 \le \lambda \le 1\}$, is contained in C.

Exercise 2.3. (1) Check that each C_d, $d \in \mathbb{N}$, is convex.

(2) Draw an example of a non-convex set.

Recall from linear algebra that a **hyperplane** in $\mathbb{R}^d$ is a set

$$H := \{\mathbf{x} \in \mathbb{R}^d : a_1 x_1 + a_2 x_2 + \cdots + a_d x_d = b\}$$