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DEVELOPMENT OF ELLIPTIC FUNCTIONS ACCORDING TO RAMANUJAN

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This book is dedicated to the memory of V. Ramaswami Iyer, the founder of the Indian Mathematical Society (I.M.S); Prof. A. Narasinga Rao, one of the earliest members of the I.M.S.; and my revered teacher Dr. B. S. Madhava Rao of the University of Mysore who passed away recently at Bangalore. I may say that these are the noblest academic people I have known in my life.

K. Venkatachaliengar

Preface

Preface to the original work

This book is devoted to the development of elliptic functions as perceived by Srinivasa Ramanujan. Ramanujan had not seen any standard book on elliptic functions before he went to England. This is clear since he did not recognize their characteristic properties—double periodicity, addition theorem, etc.,—and the notation he uses is his own. Nowhere in his treatment do we find the familiar Legendrian k, k', K, K', E, E', or the Jacobian parameter q, or the Weierstrassian $\wp$, ζ, σ, g_2, g_3, e_1, e_2 or e_3. He uses α for k^2; z for K; Q and R instead of g_2 and g_3; and so on. It is only when he published in England his two papers: "Modular equations and approximations to π" [90], and "On certain arithmetical functions" [91], he uses the modern notations and expresses his parameters in terms of the classical ones. He proves his basic identity in [91, (17)] (see (1.5)), modestly claims that the results developed there really do belong to the theory of elliptic functions, and draws the interest of the reader to the simplicity of his proofs. One finds in his paper [91] the simplest proof of the factorization of the discriminant. First, if

$$P = 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}, \quad Q = 1 + 240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}, \quad R = 1 - 504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}$$

and $|q| < 1$, then we have Ramanujan's differential equations:

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad \text{and} \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}.$$

Ramanujan used these differential equations to derive the factorization formula:

$$Q^3 - R^2 = 1728q \prod_{j=1}^{\infty} (1 - q^j)^{24}.$$

The proof of Ramanujan's differential equations is by use of the basic Ramanujan identity (1.5) which is entirely algebraic in nature. Ramanujan's entire development is also algebraic in character.

A generalized form of Ramanujan's identity—see (1.9) and (1.13)—can be guessed from his work and its proof is almost immediate. This enables us to derive the addition theorem and differential equation of the classical function $\wp(\theta)$ of Weierstrass as well as the proof of Ramanujan's differential equations mentioned above. A further generalization—see (3.17)—yields the corresponding results for the Jacobian elliptic functions.

As is well known, the theory of elliptic functions contains proofs of some famous and beautiful identities. All of these can be derived in a simple way following the work of Ramanujan.

The most intricate part of Ramanujan's work in this theory is the modular equations given in various forms, and the evaluation of the corresponding singular moduli. Ramanujan has not indicated the proofs of these, especially for the latter; only bare outlines of the findings of modular equations of lower degree are sketched in his notebooks. In Chapter 7, we will give proofs of the modular equations of degrees 3, 5, 7, 11 and 23 based solely on the quadratic transformation of Legendre's modular function λ.

Dr. V. R. Thiruvenkatachar, my lifelong friend and colleague, has cooperated with me in the entire work concerning this book. My sincere thanks to him.

I am grateful to the authorities of the Madurai Kamaraj University, especially the Vice Chancellor Prof. S. Krishnaswamy, Professors K. R. Nagarajan and T. Soundararajan, and Drs. T. V. S. Jagannathan and R. Bhaskaran, who have spared no effort to bring out the book in a very nice form.

I hope that the publication of this book will revive interest in the study of the theory of elliptic functions in the Ramanujan way. Many of the results given here with simple proofs can be used in the teaching of function theory.

K. Venkatachaliengar

Preface to the revised version

The monograph "Development of Elliptic Functions according to Ramanujan", by K. Venkatachaliengar [105] was originally published as Technical Report 2 by Madurai Kamaraj University in February, 1988. In a letter to the author, A. Weil wrote [109]:

I can well appreciate the difficulty and value of writing such a book, and I am sure the mathematical world will be grateful to you for having written it. ... it clearly reflects much work and a deep familiarity with the subject, and the points you make in connection with Ramanujan seem to me entirely convincing.

In his review of the monograph, B. C. Berndt remarked [13]:

The author has studied Ramanujan's papers and notebooks over a period of several decades. His keen insights, beautiful new theorems, and elegant proofs presented in this monograph will enrich readers ....

The views expressed by Weil and Berndt are shared by many mathematicians, so it is worthwhile to publish a corrected and edited version of the monograph with complete proofs and references.

My philosophy in preparing the revised version has been to follow Venkatachaliengar's methods and to preserve the ideas from the original monograph as much as possible. Where gaps have been found, every effort has been made to complete the proof. Occasionally it has been necessary to give an alternative proof along different lines. To bring the work up to date, additional comments have been included in notes at the end of each chapter. References, both to 19th century works as well as to recent publications, have been vigorously sought and recorded in the bibliography.

The first three chapters as far as Sec. 3.3 follow the original monograph closely. Several details have had to be worked out in order to complete the proofs in Sec. 3.4. Perhaps the biggest sticky point in the book is the unmotivated definition of x and z by (3.57) and (3.58): the proofs are correct, but the formulas must be known in advance. Even with this disadvantage, the development that follows is interesting and deserves to be studied. Some comments that may be used to motivate these formulas have been provided in the notes at the end of Chapter 3. Each of Secs. 5.4–5.6 begin with formulas that must be known in advance. References to other methods are given in the notes at the end of Chapter 5.

Another issue that required a lot of attention was the reconciliation of the results from Secs. 3.4 and 4.4 that occurs in Sec. 5.1. The reader is referred to the notes at the end of Chapter 5 for further references to alternative methods.

The title of Chapter 7 was originally "Picard's Theorem", but the chapter is really about the modular function λ, so it has been renamed. Some of the material in Sec. 7.1 has been reworked completely. I could not fix Venkatachaliengar's proof that $|a_n| < 41^n$, so I used similar ideas to give a different proof of the stronger result $|a_n| \leq 16^n$ given by (7.8), instead. It is well-known that the properties of the modular function λ can be used to prove Picard's theorem. The details given by Venkatachaliengar in [105, pp. 120–123] rely on selectively choosing branches of the square-root function to erroneously deduce a mapping property of an analytic function. This result could not be fixed, so a proof of Picard's theorem is not given in this revised version. Instead, the reader is referred to Venkatachaliengar's paper [103] for another proof of Picard's theorem, or to one of the many texts on complex analysis for the standard proof. For example, a clear and succinct deduction of Picard's theorem from properties of the modular function λ is given as an exercise in [29, p. 119].

The original monograph [105] has been reviewed by Berndt [13] and Cooper [48]. Parts of the monograph have been surveyed and extended in the works of S. Bhargava [27], Cooper [44] and Venkatachaliengar [106].

The following notation will be used throughout. Let τ be a complex number with positive imaginary part, and let h and q be defined by

$$h = \exp(i\pi\tau) \quad \text{and} \quad q = \exp(2\pi i\tau)$$

so that $q = h^2$.

In the early 1990's my teacher R. A. Askey gave me a copy of Venkatachaliengar's monograph and suggested I edit it. B. C. Berndt encouraged me to complete the editing, and always enquired as to how it was going whenever we met at a conference. Venkatachaliengar's former Ph.D. student V. Ramamani kindly took me by auto-rickshaw to meet Venkatachaliengar at his house in Bangalore on July 4, 2000. Perhaps the most ardent supporter of Venkatachaliengar's work is his grandson G. N. Srinivasa Prasanna. The conference that Prasanna organized in Bangalore as a tribute to the life and work of Venkatachaliengar during July 1–5, 2009, provided the final stimulus to complete the editing of the monograph. The conference proceedings [8] contain information about Venkatachaliengar's life and work. Berndt and H. H. Chan offered valuable comments on a draft of the revised manuscript. To all of the people mentioned in this paragraph, I offer my heartfelt thanks.

Shaun Cooper, Auckland, April 2011

Contents

Chapter 1

The Basic Identity

1.1 Introduction

Ramanujan has indicated how one can, in a simple, purely algebraic way, develop the basic properties of the classical elliptic and other allied functions. Professor Birch [28] of the University of Oxford considers Ramanujan's paper [91] to be *"one of the most beautiful papers published by this (Cambridge Philosophical) Society"*.

The title of Ramanujan's paper—"On Certain Arithmetical Functions" —does not reveal the rich and manifold contents. The first three sections of the paper contain a few asymptotic formulas concerning the divisor sums $\sigma_r(n) = \sum_{d|n} d^r$ and a connected convolution function. Sections 4–10 deal with elliptic functions, more particularly the Weierstrass elliptic function expressed by its Fourier development. In Sec. 11, Ramanujan obtains closed formulas for several connected convolution functions. Section 12 contains the beautiful identity [91, (59)]:

$$\frac{1^5 q}{1-q} + \frac{3^5 q^2}{1-q^3} + \frac{5^5 q^3}{1-q^5} + \cdots = Q \times \left(\frac{q}{1-q} + \frac{3q^2}{1-q^3} + \frac{5q^3}{1-q^5} + \cdots \right)$$

where

$$Q = 1 + 240 \left(\frac{1^3 q}{1-q} + \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1-q^3} + \cdots \right),$$

as well as a nice deduction of a convolution property for divisor sums. Here, and throughout this book, q is a complex variable that satisfies $|q| < 1$.

Let

$$f(q) = q^{1/24}(1-q)(1-q^2)(1-q^3)\cdots .$$

Ramanujan determines orders of several arithmetical functions using the four interesting formulas [91, (65)]:

$$f(q) = q^{1^2/24} - q^{5^2/24} - q^{7^2/24} + q^{11^2/24} + \cdots, \tag{1.1}$$

$$f^3(q) = q^{1^2/8} - 3q^{3^2/8} + 5q^{5^2/8} - 7q^{7^2/8} + \cdots, \tag{1.2}$$

$$\frac{f^5(q)}{f^2(q^2)} = q^{1^2/24} - 5q^{5^2/24} + 7q^{7^2/24} - 11q^{11^2/24} + \cdots, \tag{1.3}$$

$$\frac{f^5(q^2)}{f^2(-q)} = q^{1^2/3} - 2q^{2^2/3} + 4q^{4^2/3} - 5q^{5^2/3} + \cdots, \tag{1.4}$$

where 1, 5, 7, 11, ..., are the positive odd numbers without multiples of 3, and 1, 2, 4, 5, ..., are the positive integers without multiples of 3. The identities (1.1) and (1.2) are classical and are due to Euler and Jacobi, respectively. The other two identities (1.3) and (1.4) are due to Ramanujan himself. Although Ramanujan gives the exponents explicitly, Watson [107, p. 148] overlooks this and doubts whether Ramanujan was aware of their proofs. Proofs of these two formulas of Ramanujan that use the quintuple product identity will be given at the end of this book in Appendix B.

Let $\tau(n)$ be Ramanujan's tau-function defined by

$$q\prod_{j=1}^{\infty}(1-q^j)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

At the end of the paper Ramanujan gives the beautiful factorization [91, (101)]:

$$\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}$$

and makes the famous conjecture [91, (105)]:

$$|\tau(n)| \le n^{11/2}d(n),$$

where $d(n)$ denotes the number of divisors of n. The conjecture was proved half a century later by Deligne [51], using the advanced methods of algebraic geometry.

In [91, (17)] Ramanujan has given the following identity:

$$\left(\frac{1}{4}\cot\frac{\theta}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin n\theta\right)^2 \tag{1.5}$$

$$= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\cos n\theta + \frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(1-\cos n\theta).$$

He gives a completely algebraic proof of this identity. He also gives the identity [91, (18)]:

$$\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}+\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}(1-\cos n\theta)\right)^2 \tag{1.6}$$
$$=\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}(5+\cos n\theta),$$

and hints that a similar, algebraic proof may be given. The identity (1.6) corresponds to the differential equation satisfied by the Weierstrass elliptic function $\wp(z)$:

$$\wp''(z)=6\wp^2(z)-g_2/2.$$

Ramanujan obtains many interesting results from the basic identity (1.5), and these are connected with several interesting results in his notebooks. From these identities the basic properties of elliptic functions can be derived in a purely algebraic way without making any use of the Cauchy-Liouville methods of theory of functions. Ramanujan [91, Sec. 7] remarks in a very modest way: *"The elementary proof of these formulæ given in the preceding sections seems to be of some interest in itself"*.

Also, Ramanujan gives the simplest proof of the classical identity:

$$Q^3-R^2=1728q\prod_{j=1}^{\infty}(1-q^j)^{24} \tag{1.7}$$

where

$$Q=1+240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}\qquad\text{and}\qquad R=1-504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}.$$

In this connection, Ramanujan obtains his important differential equations

$$q\frac{dP}{dq}=\frac{P^2-Q}{12},\qquad q\frac{dQ}{dq}=\frac{PQ-R}{3}\quad\text{and}\quad q\frac{dR}{dq}=\frac{PR-Q^2}{2},$$

where

$$P=1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}.$$

Ramanujan's simple proof of the product formula (1.7) has escaped the attention of scholars. Weil [108, p. 34] and Lang [80, p. 249] base their proofs on the classical identities of Jacobi-Weierstrass theory. Serre, in his excellent monograph [97, p. 95], has repeated Hurwitz's well-known *a priori* proof. All these make use of Cauchy function theory.

A search through older literature reveals that Ramanujan's identity (1.5) was given by Jordan [75, p. 521, eq. (8)] in a somewhat different form. This is obtained after a good deal of elliptic function theory has been developed. Ramanujan, however, makes this identity the foundation of his theory and gives a simple *a priori* proof.

1.2 The generalized Ramanujan identity

The identity (1.5) may be generalized into a form whose proof becomes very simple. Let

$$\rho_1(z) = \frac{1}{2} + \sum_n{}' \frac{z^n}{1-q^n} \quad \text{and} \quad \rho_2(z) = -\frac{1}{12} + \sum_n{}' \frac{q^n z^n}{(1-q^n)^2}, \tag{1.8}$$

where the symbol $\sum_n{}'$ denotes that the summations are over all non-zero integers n. By the ratio test, the series defining ρ_1 converges for $|q| < |z| < 1$, while the series defining ρ_2 converges for $|q| < |z| < |q|^{-1}$. The generalized Ramanujan identity is the following: For any three complex numbers α, β and γ satisfying $\alpha\beta\gamma = 1$ and $|q| < |\alpha|, |\beta|, |\alpha\beta| < 1$, we have

$$\rho_1(\alpha)\rho_1(\beta) - \rho_1(\alpha\beta)(\rho_1(\alpha) + \rho_1(\beta)) = \rho_2(\alpha) + \rho_2(\beta) + \rho_2(\gamma). \tag{1.9}$$

This may be proved as follows. Let the Laurent expansion of the left hand side of (1.9) be

$$\sum_{m,n=-\infty}^{\infty} c_{m,n}\alpha^m\beta^n.$$

Substituting the expansions of $\rho_1(\alpha)$, $\rho_1(\beta)$ and $\rho_1(\alpha\beta)$ in the left hand side of (1.9) we evaluate the coefficients $c_{m,n}$ distinguishing four separate cases.

Case (i): $mn(m-n) \neq 0$. Then using (1.9) we obtain

$$c_{m,n} = \frac{1}{(1-q^m)(1-q^n)} - \frac{1}{(1-q^n)(1-q^{m-n})} - \frac{1}{(1-q^m)(1-q^{n-m})} = 0.$$

Case (ii): If $m \neq 0$ and $n = 0$ then

$$c_{m,0} = \frac{1}{2(1-q^m)} - \frac{1}{2(1-q^m)} - \frac{1}{(1-q^m)(1-q^{-m})} = \frac{q^m}{(1-q^m)^2}.$$

Similarly, if $m = 0$ and $n \neq 0$, then $c_{0,n} = \dfrac{q^n}{(1-q^n)^2}$.

Case (iii): $m = n \neq 0$. Then

$$c_{m,m} = \frac{1}{(1-q^m)^2} - \frac{1}{(1-q^m)} = \frac{q^m}{(1-q^m)^2}.$$

Case (iv): $m = n = 0$. Then $c_{0,0} = 1/4 - 1/2 = -1/4$.

The right hand side of (1.9) consists of three terms besides the constant term, whose coefficients are easily seen to be identical with the corresponding terms of the left hand side that arise from Cases (ii) and (iii). The significant thing we observe is that the bulk of the terms on the left hand side, namely those with $mn(m-n) \neq 0$, cancel out. This completes the proof of the identity (1.9).

We now transform the functions ρ_1 and ρ_2 into appropriate forms which extend the range of their definition. For the function ρ_1, we have:

$$\begin{aligned} \rho_1(z) &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{z^m}{1-q^m} + \sum_{m=1}^{\infty} \frac{z^{-m}}{1-q^{-m}} \qquad (1.10) \\ &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{z^m(1-q^m+q^m)}{1-q^m} - \sum_{m=1}^{\infty} \frac{z^{-m}q^m}{1-q^m} \\ &= \frac{1}{2} + \frac{z}{1-z} + \sum_{m=1}^{\infty} \frac{q^m}{1-q^m}(z^m - z^{-m}). \end{aligned}$$

This can be expanded as a double series:

$$\rho_1(z) = \frac{1+z}{2(1-z)} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(z^m q^{mn} - z^{-m}q^{mn}).$$

We interchange the order of summation in the above and obtain

$$\rho_1(z) = \frac{1+z}{2(1-z)} + \sum_{n=1}^{\infty}\left(\frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}\right). \qquad (1.11)$$

This shows that ρ_1 is analytic in $0 < |z| < \infty$ except at $z = q^r$, $r = 0, \pm 1, \pm 2, \ldots$, where there are simple poles. There is an essential singularity at $z = 0$. From (1.11) we may deduce the symmetric property

$$\rho_1(z) + \rho_1(z^{-1}) = 0,$$

and the quasi-periodic property

$$\rho_1(z) - \rho_1(qz) = -1.$$