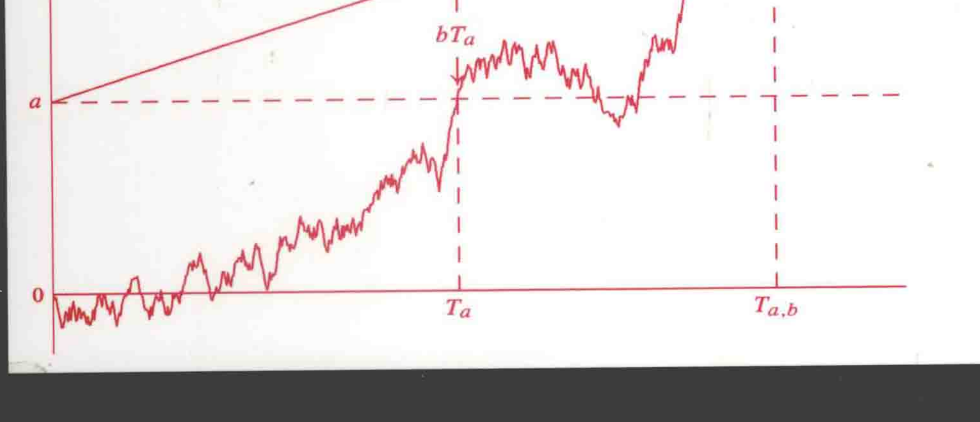
Douglas Kennedy

Stochastic Financial Models



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Stochastic Financial Models

Douglas Kennedy

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Preface

The bulk of this work originated in lecture notes prepared for two courses that I introduced for students of mathematics at the University of Cambridge, dating back to 1992, and with which I was associated for fifteen years. The first of these was for final-year undergraduates and it covered the material in Chapter 1 and roughly half of each of Chapters 2, 4 and 5. The second was for first-year graduate students and it dealt with most of Chapters 2–6 at an accelerated pace; it did not have the former course as a prerequisite.

While students were expected to have a good prior knowledge of elementary probability theory and perhaps some acquaintance with Markov chains, no background in measure-theoretic probability was assumed for either course, although in each case students had the opportunity to take a course on that topic concurrently; for the more advanced option, an introductory course on stochastic calculus was also available to be taken at the same time. Apart from the intrinsic interest of presenting the material on mathematical finance, a major pedagogical motivation for introducing the courses was to stimulate students to learn more about probability, martingales and stochastic integration by exposing them to one of the most important and exciting areas of application of those topics.

The introduction to mathematical finance presented here is designed to slot in between those works that provide a survey of the field with a relatively light mathematical content and those books at the other end of the spectrum, which take no prisoners in their rigorous, formal approach to stochastic integration and probabilistic ideas. In many places in the book the slant is toward a classical applied mathematical approach with a concentration on calculations rather than necessarily seeking the greatest generality. To avoid breaking the flow of material, where concepts from measure-theoretic probability are required, for the most part they are not introduced in the main body of the text but have been gathered in the mathematical preliminaries in Appendix A; the reader is also encouraged to consult the books suggested for further reading. To assist self study, solutions to all the exercises are given in Appendix B but students are urged strongly to attempt the problems unaided before consulting the solutions.

It is not necessary to follow the material in the book in a strict linear order. To provide some route maps through the chapters, it should be noted that the material in Chapter 1 is orthogonal to much of the remaining book in that it deals with the more classical topics of utility and the mean-variance approach to portfolio choice, rather than being concerned with derivative pricing, which is the focus in the remainder of the book. If the existence of an individual utility function is taken as given, then this chapter is not required for an understanding of the subsequent material but the chapter is included to give a more rounded view of finance generally. A full understanding of the binomial model, presented in Chapter 2, is central for getting to grips with the pricing of derivatives by self-financing hedging portfolios. It should be possible for the reader to proceed directly from this chapter to the Black–Scholes model in Chapter 5 without studying the general discrete-time model in Chapter 3, having acquired sufficient background on Brownian motion from Chapter 4. For example, if the reader wanted to get quickly to the Black–Scholes formula at a first reading, it would be possible to omit consideration of hitting-time distributions for Brownian motion in Section 4.2, and with the ideas of Sections 4.1, 4.3 and 4.4, proceed to study Sections 5.1 and 5.2; then Section 4.5 on stochastic calculus could be consulted to give the basis for reading Section 5.3 on hedging in the Black–Scholes context. One might then return to look at hitting-time distributions before dealing with path-dependent options in Section 5.4.

I must thank all the former students whose helpful observations contributed to the development of this material, but I am particularly grateful to Bryn Thompson-Clarke and Wenjie Xiang, who gave insightful comments on an early draft of the book, and also to Rob Calver of Chapman & Hall for his encouragement during the project. I am also indebted to an anonymous reviewer who provided very useful and perceptive suggestions. Finally, I want to thank all those in Trinity College who have contributed to making such a congenial, stimulating and beautiful working environment, which I have been privileged to enjoy for thirty-five years.

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Chapter 1

PORTFOLIO CHOICE

1.1 Introduction

The contents of this chapter are somewhat different in approach to much of the remainder of the book. In subsequent chapters we deal principally with the problem of pricing derivative securities; that is, secondary contracts for which the payoff is dependent on the price of an underlying asset, such as a stock. In complete markets, as in the cases of the binomial model of Chapter 2 or the Black–Scholes model of Chapter 5, the price of any derivative contract, or contingent claim, is derived objectively through 'hedging'; the payoff of the contract may be duplicated exactly by trading in the underlying asset and a bank account so that there is no inherent risk to the seller of the contract. The price of the contract is just the initial cost of setting up the trading strategy that duplicates the payoff; it is objective and risk free in that two investors may have different views on how the price of the underlying asset may evolve in the future but they will agree on the price of the contract. Such ideas form the basis of much of modern financial theory.

By contrast, the two topics in this chapter deal with the attitudes of individual investors in relation to investment decisions where these decisions are subjective in nature. The first is the notion of an investor's individual utility function. In a deterministic model it is reasonable to expect that an investor will seek to choose an investment portfolio of assets in order to maximize the final wealth that he achieves. When the model is stochastic the investor's final wealth will typically be a random variable, W, and it would no longer make sense for him to make investment decisions seeking to maximize a random quantity. Instead, he might wish to maximize the expected value of his final wealth, $\mathbb{E}(W)$, so that he achieves the largest wealth on average, or more generally it is often postulated that he seeks to maximize $\mathbb{E}\, v(W)$ for some appropriate function $v(\cdot)$; this function is referred to as the investor's utility function. It is individual to the investor and we show in Section 1.2 that any investor who orders his preferences of random outcomes in a suitably consistent way possesses an essentially unique utility function and that properties of this function may characterize his attitude towards risk.

The second topic of the chapter in Section 1.3 is concerned with mean-variance analysis where, among portfolios giving a fixed mean return, an investor chooses the portfolio with smallest variance of the return. The model is subjective both in its choice of optimality criterion but also in its dependence on the investors beliefs about the means of the returns of the various available assets as well as the covariances between those returns. The capital-asset pricing model in Section 1.3.5 considers the implications for the whole market of the actions of individual investors.

The material of this chapter represents a significant step in the development of mathematical models in finance; its importance was recognized by the award of the Nobel Prize in Economics in 1990 to Harry Markowitz, for his contributions to the theory of portfolio choice, and to William Sharpe, for his work on the capital-asset pricing model.

1.2 Utility

1.2.1 Preferences and utility

We begin by discussing the classical justification for assuming that an investor who may order his preferences for investments in a consistent manner has an essentially unique utility function; furthermore, the properties of this utility function characterize his attitude to risk. We outline the axiomatic approach to showing the existence of such a utility function when it is assumed that the investor's preferences satisfy certain axioms.

Let Γ be a sample space representing the set of possible outcomes of some gambles with random payoffs. Let $\mathcal{P}$ be a set of probabilities on Γ, so that an element $\mathbb{A} \in \mathcal{P}$ is a real-valued function defined on subsets (events) of Γ satisfying the following three conditions:

1. $0 \leqslant \mathbb{A}(G) \leqslant 1$, for all $G \subseteq \Gamma$;
2. $\mathbb{A}(\Gamma) = 1$; and
3. for a finite or infinite collection of disjoint events $\{G_i\}_i$, that is $G_i \cap G_j = \emptyset$ for $i \neq j$, we have
$$\mathbb{A}\left(\bigcup_i G_i\right) = \sum_i \mathbb{A}(G_i).$$

We refer to an element $\mathbb{A} \in \mathcal{P}$ as a **gamble** where $\mathbb{A}$ may be thought of as the probability distribution of the outcome of the gamble. We assume that the set $\mathcal{P}$ is closed under convex combinations so that for any $\mathbb{A}, \mathbb{B} \in \mathcal{P}$ and $0 \leqslant p \leqslant 1$ we assume that $p\mathbb{A} + (1-p)\mathbb{B} \in \mathcal{P}$. The gamble $p\mathbb{A} + (1-p)\mathbb{B}$ takes the value $p\mathbb{A}(G) + (1-p)\mathbb{B}(G)$ for events $G \subseteq \Gamma$ and it is of course a probability on Γ; this gamble would correspond to the situation where the investor tosses a coin with probability p of 'heads' and $1-p$ of 'tails' and chooses gamble $\mathbb{A}$ or gamble $\mathbb{B}$ according to whether the outcome is heads or tails. It is an immediate consequence of this assumption that for any gambles $\mathbb{A}_1, \ldots, \mathbb{A}_k \in \mathcal{P}$ and for real numbers $p_i \geqslant 0$, $1 \leqslant i \leqslant k$ with $\sum_{i=1}^k p_i = 1$, by induction on k we see that

$$p_1\mathbb{A}_1 + \cdots + p_k\mathbb{A}_k \in \mathcal{P}.$$

We will assume that an investor (or gambler) has a preference relation $\succ$ on $\mathcal{P}$; this corresponds to some given subset $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ with $\mathbb{A} \succ \mathbb{B}$ if and only if $(\mathbb{A}, \mathbb{B}) \in \mathcal{S}$ for gambles $\mathbb{A}, \mathbb{B} \in \mathcal{P}$. Read $\mathbb{A} \succ \mathbb{B}$ as $\mathbb{A}$ **is preferred to** $\mathbb{B}$. This defines a relation $\sim$ on $\mathcal{P}$ by setting $\mathbb{A} \sim \mathbb{B}$ when $\mathbb{A} \not\succ \mathbb{B}$ and $\mathbb{B} \not\succ \mathbb{A}$ for $\mathbb{A}, \mathbb{B} \in \mathcal{P}$, that is when $(\mathbb{A}, \mathbb{B}) \notin \mathcal{S}$ and $(\mathbb{B}, \mathbb{A}) \notin \mathcal{S}$. We will refer to $\sim$ as an indifference relation and say that the investor is **indifferent between** $\mathbb{A}$ and $\mathbb{B}$ when $\mathbb{A} \sim \mathbb{B}$. We will assume here that the relations $\succ$ and $\sim$ satisfy some plausible axioms which would imply rational consistency on the part of the investor in ordering his preferences.

Axioms

1. For any $\mathbb{A}, \mathbb{B} \in \mathcal{P}$ exactly one of the following holds:

   (i) $\mathbb{A} \succ \mathbb{B}$; (ii) $\mathbb{B} \succ \mathbb{A}$; or (iii) $\mathbb{A} \sim \mathbb{B}$.

2. The relation $\sim$ is an equivalence relation on $\mathcal{P}$; that is,

   (i) $\mathbb{A} \sim \mathbb{A}$ for all $\mathbb{A} \in \mathcal{P}$;
   (ii) for any $\mathbb{A}, \mathbb{B} \in \mathcal{P}$, if $\mathbb{A} \sim \mathbb{B}$ then $\mathbb{B} \sim \mathbb{A}$; and
   (iii) for any $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$, if $\mathbb{A} \sim \mathbb{B}$ and $\mathbb{B} \sim \mathbb{C}$ then $\mathbb{A} \sim \mathbb{C}$.

3. For any $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$, if $\mathbb{A} \succ \mathbb{B}$ and $\mathbb{B} \succ \mathbb{C}$ then $\mathbb{A} \succ \mathbb{C}$.

4. For any $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$,

   (i) if $\mathbb{A} \succ \mathbb{B}$ and $\mathbb{B} \sim \mathbb{C}$ then $\mathbb{A} \succ \mathbb{C}$; and
   (ii) if $\mathbb{A} \sim \mathbb{B}$ and $\mathbb{B} \succ \mathbb{C}$ then $\mathbb{A} \succ \mathbb{C}$.

5. For any $\mathbb{A}, \mathbb{C} \in \mathcal{P}$ and $p \in [0, 1]$, if $\mathbb{A} \sim \mathbb{C}$ and $\mathbb{B} \in \mathcal{P}$ then $p\mathbb{A} + (1 - p)\mathbb{B} \sim p\mathbb{C} + (1 - p)\mathbb{B}$.

6. For any $\mathbb{A}, \mathbb{C} \in \mathcal{P}$ and $p \in (0, 1]$, if $\mathbb{A} \succ \mathbb{C}$ and $\mathbb{B} \in \mathcal{P}$ then $p\mathbb{A} + (1-p)\mathbb{B} \succ p\mathbb{C} + (1 - p)\mathbb{B}$.

7. For any $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$, if $\mathbb{A} \succ \mathbb{C} \succ \mathbb{B}$ then there exists $p \in [0, 1]$ with $p\mathbb{A} + (1 - p)\mathbb{B} \sim \mathbb{C}$.

We observe first that the p in Axiom 7 is unique.

Lemma 1.1 *Suppose that $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{P}$ with $\mathbb{A} \succ \mathbb{C} \succ \mathbb{B}$ and $p\mathbb{A} + (1-p)\mathbb{B} \sim \mathbb{C}$ then $0 < p < 1$ and p is unique.*

Proof. Trivially $p \neq 0$ or 1. Suppose that p is not unique so that there exists q with $q\mathbb{A} + (1 - q)\mathbb{B} \sim \mathbb{C}$. Without loss of generality assume that $q < p$ so that we have $0 < p - q < 1 - q$. But

$$\mathbb{B} = \left(\frac{p-q}{1-q}\right) \mathbb{B} + \left(\frac{1-p}{1-q}\right) \mathbb{B} \quad \text{and} \quad \mathbb{A} \succ \mathbb{B},$$

then by Axiom 6

$$\left(\frac{p-q}{1-q}\right)\mathbb{A} + \left(\frac{1-p}{1-q}\right)\mathbb{B} \succ \mathbb{B}.$$

However

$$p\mathbb{A} + (1-p)\mathbb{B} = q\mathbb{A} + (1-q)\left[\left(\frac{p-q}{1-q}\right)\mathbb{A} + \left(\frac{1-p}{1-q}\right)\mathbb{B}\right],$$

and by Axiom 6 again, this implies that

$$p\mathbb{A} + (1-p)\mathbb{B} \succ q\mathbb{A} + (1-q)\mathbb{B}$$

which gives a contradiction. □

We may now establish the existence and linearity of a function which quantifies the preferences when those preferences are formulated consistently in that they obey the Axioms.

Theorem 1.1 *There exists a real-valued function $f : \mathcal{P} \to \mathbb{R}$ with*

$$f(\mathbb{A}) > f(\mathbb{B}) \quad \text{if and only if} \quad \mathbb{A} \succ \mathbb{B}, \tag{1.1}$$

and

$$f(p\mathbb{A} + (1-p)\mathbb{B}) = pf(\mathbb{A}) + (1-p)f(\mathbb{B}) \tag{1.2}$$

for any $\mathbb{A}, \mathbb{B} \in \mathcal{P}$ and $0 \leqslant p \leqslant 1$. Furthermore, f is unique up to affine transformations; that is, if g is any other such function satisfying (1.1) *and* (1.2) *then there exist real numbers $\alpha > 0$ and β with $g(\mathbb{A}) = \alpha f(\mathbb{A}) + \beta$, for all $\mathbb{A} \in \mathcal{P}$.*

Proof. If we have $\mathbb{A} \sim \mathbb{B}$ for all $\mathbb{A}, \mathbb{B} \in \mathcal{P}$ then take $f(\mathbb{A}) \equiv 0$ and the conclusions are immediate. So suppose that there exists a pair $\mathbb{C}, \mathbb{D} \in \mathcal{P}$ with $\mathbb{C} \succ \mathbb{D}$. By the axioms, for any $\mathbb{A} \in \mathcal{P}$ there are five possibilities: (a) $\mathbb{A} \succ \mathbb{C}$, (b) $\mathbb{A} \sim \mathbb{C}$, (c) $\mathbb{C} \succ \mathbb{A} \succ \mathbb{D}$, (d) $\mathbb{A} \sim \mathbb{D}$ and (e) $\mathbb{D} \succ \mathbb{A}$. First define $f(\mathbb{C}) = 1$ and $f(\mathbb{D}) = 0$. We define $f(\mathbb{A})$ for $\mathbb{A}$ satisfying each case in turn. For case (a) there exists a unique $p \in (0, 1)$ with $p\mathbb{A} + (1-p)\mathbb{D} \sim \mathbb{C}$; define $f(\mathbb{A}) = 1/p$. For case (b) set $f(\mathbb{A}) = 1$. For case (c) there exists a unique $q \in (0, 1)$ with $q\mathbb{C} + (1-q)\mathbb{D} \sim \mathbb{A}$; define $f(\mathbb{A}) = q$. For case (d) set $f(\mathbb{A}) = 0$. Finally, for case (e) there exists a unique $r \in (0, 1)$ with $r\mathbb{C} + (1-r)\mathbb{A} \sim \mathbb{D}$ and define $f(\mathbb{A}) = -r/(1-r)$.

To check that f satisfies (1.1) and (1.2) for all $\mathbb{A}$ and $\mathbb{B}$ requires checking fifteen different cases for $\mathbb{A}$ and $\mathbb{B}$; these correspond to the five instances where both $\mathbb{A}$ and $\mathbb{B}$ satisfy one of the five cases (a)-(e), together with the $\binom{5}{2} = 10$ instances when $\mathbb{A}$ and $\mathbb{B}$ are in different cases of (a)-(e).

We give the details in just one situation when both $\mathbb{A}$ and $\mathbb{B}$ are in case (c) so that $\mathbb{C} \succ \mathbb{A} \succ \mathbb{D}$ and $\mathbb{C} \succ \mathbb{B} \succ \mathbb{D}$. We have $f(\mathbb{A}) = q_1$ and $f(\mathbb{B}) = q_2$, say, where

$$\mathbb{A} \sim q_1\mathbb{C} + (1-q_1)\mathbb{D} \quad \text{and} \quad \mathbb{B} \sim q_2\mathbb{C} + (1-q_2)\mathbb{D}.$$

When $q_1 = q_2$ then $\mathbb{A} \sim \mathbb{B}$ and (1.1) holds. When $q_1 > q_2$ then, as in the proof of Lemma 1.1,

$$q_1\mathbb{C} + (1 - q_1)\mathbb{D} \succ q_2\mathbb{C} + (1 - q_2)\mathbb{D}$$

and thus $\mathbb{A} \succ \mathbb{B}$ giving (1.1); similarly, when $q_1 < q_2$ it follows that $\mathbb{B} \succ \mathbb{A}$. To see that (1.2) holds, let $p \in (0, 1)$ and then by Axiom 5

$$p\mathbb{A} + (1 - p)\mathbb{B} \sim [p\,(q_1\mathbb{C} + (1 - q_1)\mathbb{D}) + (1 - p)\,(q_2\mathbb{C} + (1 - q_2)\mathbb{D})]$$

which may be rearranged to show that

$$p\mathbb{A} + (1 - p)\mathbb{B} \sim [(pq_1 + (1 - p)q_2)\,\mathbb{C} + (p(1 - q_1) + (1 - p)(1 - q_2))\,\mathbb{D}]\,.$$

It follows from the definition of f that

$$f(p\mathbb{A} + (1 - p)\mathbb{B}) = pq_1 + (1 - p)q_2 = pf(\mathbb{A}) + (1 - p)f(\mathbb{B}),$$

which establishes (1.2) in this case.

To verify that f is unique up to affine transformations, suppose that g is any other function satisfying (1.1) and (1.2). Because $\mathbb{C} \succ \mathbb{D}$ we must have $g(\mathbb{C}) > g(\mathbb{D})$, then define $\beta = g(\mathbb{D})$ and $\alpha = g(\mathbb{C}) - g(\mathbb{D}) > 0$. Now suppose that $\mathbb{A}$ is in case (c) so that $\mathbb{C} \succ \mathbb{A} \succ \mathbb{D}$. If $f(\mathbb{A}) = q$ then $\mathbb{A} \sim q\mathbb{C} + (1 - q)\mathbb{D}$ and it follows that

$$\begin{aligned} g(\mathbb{A}) &= g(q\mathbb{C} + (1 - q)\mathbb{D}) = qg(\mathbb{C}) + (1 - q)g(\mathbb{D}) \\ &= q(\alpha + \beta) + (1 - q)\beta = q\alpha + \beta = \alpha f(\mathbb{A}) + \beta. \end{aligned}$$

The other cases follow in a similar fashion. □

This result establishes that for an investor with a consistent set of preferences there exists a function f, unique up to affine transformations, which quantifies the ordering of his preferences in the sense of (1.1). Note that it is an immediate consequence of (1.2) that for gambles $\mathbb{A}_1, \ldots, \mathbb{A}_k \in \mathcal{P}$ and $p_i \geqslant 0$, $1 \leqslant i \leqslant k$, with $\sum_{i=1}^{k} p_i = 1$ the function f satisfies

$$f\left(\sum_{i=1}^{k} p_i \mathbb{A}_i\right) = \sum_{i=1}^{k} p_i f(\mathbb{A}_i); \tag{1.3}$$

this is established by induction on k.

Suppose that $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ has only a finite number of outcomes. Let $\mathbb{A}_i$ be the probability that assigns 1 to the outcome γ_i, $i = 1, \ldots, n$, and 0 to the other outcomes and assume that $\mathbb{A}_i \in \mathcal{P}$ for each i. Let $\mathbb{A} = (p_1, \ldots, p_n)$ be the probability distribution assigning the probability p_i to γ_i where $p_i \geqslant 0$ and $\sum_{i=1}^{n} p_i = 1$. From (1.3), it follows that

$$f(\mathbb{A}) = \sum_{i=1}^{n} p_i f(\mathbb{A}_i)$$

so that $f(\mathbb{A})$ is the expected value of the random variable which takes the value $f(\mathbb{A}_i)$ when the outcome is γ_i.

To put these ideas into the context in which they are typically encountered in finance, consider an investor who is faced with a range of investments each of which yields a payoff which is a real-valued random variable defined on some underlying probability sample space Ω, which is equipped with a probability (measure) $\mathbb{P}$. Let $\mathcal{X}$ be the set of real-valued random variables defined on Ω and, for each random variable $X \in \mathcal{X}$, let $\mathbb{P}^X$ denote the probability distribution on $\mathbb{R}$ induced by X. Here we will take the sample space Γ in the above description to be $\Gamma = \mathbb{R}$.

Suppose that the investor has a preference system ($\succ$ and $\sim$) that orders the gambles (or investments) $\mathcal{P} = \{\mathbb{P}^X : X \in \mathcal{X}\}$ in a consistent way according to the Axioms, then we know that there exists a function f so that $\mathbb{P}^X \succ \mathbb{P}^Y$ (or we may write $X \succ Y$, equivalently) if and only if $f(\mathbb{P}^X) > f(\mathbb{P}^Y)$. Let us consider the case where each random variable takes on a finite number of values so that the range of X is $\mathcal{R}(X) = \{x_1, \ldots, x_m\}$, say; then for $x \in \mathbb{R}$ we have

$$\mathbb{P}^X(\{x\}) = \begin{cases} \mathbb{P}(X = x) & \text{for} \quad x \in \mathcal{R}(X), \\ 0 & \text{for} \quad x \notin \mathcal{R}(X). \end{cases}$$

For any $x \in \mathbb{R}$ denote by $\mathbb{P}^x$ the probability distribution which assigns 1 to the point x and 0 to all other points of $\mathbb{R}$ and define a function $v : \mathbb{R} \to \mathbb{R}$ by setting $v(x) = f(\mathbb{P}^x)$, for $x \in \mathbb{R}$. With this notation, the relation (1.3) is the statement that

$$f(\mathbb{P}^X) = \sum_{i=1}^{m} f(\mathbb{P}^{x_i})\mathbb{P}(X = x_i) = \sum_{i=1}^{m} v(x_i)\mathbb{P}(X = x_i) = \mathbb{E}\,v(X).$$

The conclusion (1.1) then becomes

$$\mathbb{E}\,v(X) > \mathbb{E}\,v(Y) \quad \text{if and only if} \quad X \succ Y. \tag{1.4}$$

The function $v(\cdot)$ is known as the investor's **utility** function; it is unique up to the affine transformation implied by Theorem 1.1; that is, it is unique up to transformations of the form $\overline{v}(x) = av(x) + b$ for constants $a > 0$ and b, and it is determined by his individual preference system. The relation (1.4) implies that when the investor is faced with a number of investments with random payoff he will choose the one with largest expected utility; in subsequent sections we will refer to an individual acting in this way as a **utility-maximizing** investor. We will see in the next section that properties of the utility function indicate details of the attitude of the investor towards risk.

The discussion that leads to (1.4) was restricted to the situation where the random variables take only finitely many values. The result may be extended to arbitrary random variables but it requires consideration of closure properties of the set of gambles and consistency of the preference Axioms under countable convex combinations of gambles.

1.2.2 Utility and risk aversion

We will assume here that the outcome of an investment is described by a random variable X (defined on some sample space Ω with probability $\mathbb{P}$) and that the preferences of an investor may be described as in the previous section by a utility function $v : \mathbb{R} \to \mathbb{R}$ with the investor preferring investments with higher expected utility. Denote by $\mathbb{E}_{\mathbb{P}}$ the expectation taken with the probability $\mathbb{P}$. We say that the investor is **risk averse** when

$$\mathbb{E}_{\mathbb{P}} v(X) \leqslant v(\mathbb{E}_{\mathbb{P}} X), \tag{1.5}$$

for all random variables X and all probabilities $\mathbb{P}$. The investor is risk averse if and only if his utility function is concave. To see this, for two fixed values $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ suppose that the probability $\mathbb{P}$ is such that $\mathbb{P}(X = x) = \lambda$ and $\mathbb{P}(X = y) = 1 - \lambda$ then (1.5) implies that

$$\lambda v(x) + (1 - \lambda)v(y) \leqslant v(\lambda x + (1 - \lambda)y)$$

for all $x, y \in \mathbb{R}$ and $0 \leqslant \lambda \leqslant 1$ which is the statement that v is concave; conversely, when v is concave then (1.5) is just Jensen's inequality. The investor being risk averse implies that he prefers a certain (that is, deterministic) outcome of μ, say, to an investment X with mean $\mathbb{E}_{\mathbb{P}} X = \mu$.

The investor is **risk neutral** when $\mathbb{E}_{\mathbb{P}} v(X) = v(\mathbb{E}_{\mathbb{P}} X)$ for all $\mathbb{P}$ and X; risk neutrality is equivalent to the utility function v being affine and it means that the investor is indifferent between a random outcome with mean μ and a certain outcome of μ.

The investor is **risk preferring** when $\mathbb{E}_{\mathbb{P}}\, v(X) \geqslant v(\mathbb{E}_{\mathbb{P}} X)$ for all $\mathbb{P}$ and X and it corresponds to the utility function v being convex.

To induce a risk-averse investor to undertake an investment with payoff X and probability $\mathbb{P}$ then a **compensatory risk premium**, α, would have to be offered where α would satisfy

$$\mathbb{E}\, v(\alpha + X) = v(\mu) \quad \text{with} \quad \mu = \mathbb{E} X.$$

We have now suppressed the dependence on the underlying probability $\mathbb{P}$ in the notation. Here the quantity α represents the (deterministic) amount that would have to be added to the payoff of a risky investment X with mean μ to make the investor indifferent between the enhanced risky investment and the certain amount μ.

A related notion is that of an **insurance risk premium**, β, defined by

$$\mathbb{E}\, v(X) = v(\mu - \beta). \tag{1.6}$$

The quantity β is the amount that the risk-averse investor would be willing to pay to avoid the 'fair' investment X with mean μ. Note that when X and Y are two investments with the same mean $\mathbb{E} X = \mathbb{E} Y = \mu$ and $v(\cdot)$ is a strictly increasing function then $X \succ Y$ if and only if $\beta_X < \beta_Y$, where β_X and β_Y are the respective insurance risk premiums; this follows because

$$v(\mu - \beta_X) = \mathbb{E}\, v(X) > \mathbb{E}\, v(Y) = v(\mu - \beta_Y)$$

if and only if $\beta_X < \beta_Y$.

When we expand on the left-hand side of (1.6) using Taylor's Theorem we have

$$\begin{aligned}\mathbb{E}\, v(X) &= \mathbb{E}\left[v(\mu) + (X-\mu)v'(\mu) + \frac{(X-\mu)^2}{2}v''(\mu) + \cdots\right] \\ &= v(\mu) + \frac{\mathbb{V}ar X}{2}v''(\mu) + \cdots\end{aligned}$$

since $\mathbb{E}\, X = \mu$. Perform a similar expansion on the right-hand side of (1.6) to see that

$$v(\mu - \beta) = v(\mu) - \beta v'(\mu) + \cdots .$$

and when we equate these two expressions, ignoring β^2 and higher-order terms in β as well as the terms $\mathbb{E}\,|X-\mu|^k$ for $k \geqslant 3$, we obtain the approximation

$$\beta \approx \frac{1}{2}\left[\frac{-v''(\mu)}{v'(\mu)}\right] \mathbb{V}ar X.$$

The quantity $-v''(\mu)/v'(\mu)$ is known as the **Arrow-Pratt absolute risk aversion**; it is a measure of how averse the investor is to any investment with mean μ. A related measure of risk aversion is the quantity $-\mathbb{E}\, v''(X)/\mathbb{E}\, v'(X)$, known as the **global absolute risk aversion** which is measuring the investor's aversion to the particular investment X.

The most important source of examples of utility functions is the class of **hyperbolic absolute risk aversion** functions (HARA functions) which have the form

$$v(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \tag{1.7}$$

for constants a, b and γ; the range of definition is for values of x for which the term $ax/(1-\gamma) + b \geqslant 0$, so usually we have $b \geqslant 0$. Note that the Arrow–Pratt absolute risk aversion for the function in (1.7) is

$$-\frac{v''(x)}{v'(x)} = \left(\frac{x}{1-\gamma} + \frac{b}{a}\right)^{-1}.$$

The following utility functions that will be used in subsequent chapters may be viewed as special cases or limiting cases of possibly affine transformations of (1.7); they are often chosen for their mathematical tractability.

(a) Quadratic: $v(x) = x - \frac{1}{2}\theta x^2$; take $\gamma = 2, a = \sqrt{\theta}, ab = 1$.

(b) Exponential: $v(x) = -e^{-ax}$; let $\gamma \to -\infty$. Note that this function has constant absolute risk aversion, a.

(c) Power: $v(x) = x^{\gamma}$ with $\gamma > 0$. Note that this is strictly concave only when $\gamma < 1$. The case $\gamma = 1$ gives the **risk-neutral** utility.

(d) Logarithmic: $v(x) = \ln x$. This follows from (1.7), by using l'Hôpital's rule to see that as $\gamma \to 0$, $(x^{\gamma} - 1)/\gamma \to \ln x$.