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NONLINEAR FUNCTIONAL ANALYSIS AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Editors

A. Ambrosetti

Scuola Normale Superiore, Pisa, Italy

K.-C. Chang

Peking University, P. R. China

I. Ekeland

Université de Paris IX, France

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FOREWORD

Since 1996 the Abdus Salam International Centre for Theoretical Physics planned to devote a series of three Schools to Nonlinear Functional Analysis and Applications to Differential Equations.

The first School was devoted to discuss the most basic tools of Nonlinear Functional Analysis. The second School, held at the Centre from 21 April to 9 May 1997, was at a higher level, combining advanced courses with a first introduction to research problems.

The present volume collects most of the lectures delivered on this occasion.

We would like to express our warm thanks to the speakers for their efforts and the high level of their lectures, and to the participants for their active role.

We are also grateful to the ICTP and the European Commission for their financial support.

Last but not least we wish to thank Professors Bertocchi and Narasimhan for having accepted our proposal and Ms. A. Bergamo and Ms. L. Zetto for their valuable secretarial assistance.

A. Ambrosetti
K.-C. Chang
I. Ekeland

CONTENTS

RECENT RESULTS FOR ASYMMETRIC NONLINEAR BOUNDARY VALUE PROBLEMS

D. ARCOYA
Departamento de Análisis Matemático, Universidad de Granada.
18071 - Granada, SPAIN
E-mail: darcoya@goliat.ugr.es

Dedicado a David Arcoya Tejada

We study the existence of solutions of asymmetric boundary value problems for equations like

$$-\Delta u = f(x,u) + h(x), \ x \in \Omega$$

with Ω being a smooth bounded domain in $\mathbb{R}^N$, $h \in L^2(\Omega)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ is a subcritical nonlinearity satisfying

$$-\infty \leq f'(-\infty) \equiv \lim_{u \to -\infty} \frac{f(x,u)}{u} < f'(+\infty) \equiv \lim_{u \to +\infty} \frac{f(x,u)}{u} \leq +\infty.$$

Classical and recent results are reviewed.

1 Introduction

In these notes we survey some new results for semilinear elliptic boundary value problems (b.v.p.) with an asymmetric nonlinear term. Specifically, we study the existence of solutions for the P.D.E. equation

$$-\Delta u = f(x,u) + h(x), \ x \in \Omega$$

under zero boundary conditions either of Dirichlet type or of Neumann type. Here, and throughout the paper, Ω is a bounded domain in $\mathbb{R}^N$ with sufficient smooth boundary $\partial\Omega$, h belongs to the Lebesgue space $L^2(\Omega)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ is a subcritical nonlinearity satisfying an asymptotic growth condition at $-\infty$ different from the one at $+\infty$ (asymmetric nonlinearity). Indeed, we assume that the following limits exist and are uniform in $x \in \Omega$:

$$-\infty \leq f'(-\infty) \equiv \lim_{u \to -\infty} \frac{f(x,u)}{u} < f'(+\infty) \equiv \lim_{u \to +\infty} \frac{f(x,u)}{u} \leq +\infty. \quad (1)$$

It is well-known that the interaction of the derivative with respect to u of the nonlinearity f with the spectrum of the Laplace operator is strongly related to the existence of solutions of these problems. Thus, we begin by discussing in Section 2, for Dirichlet boundary conditions, the classical results by Hammerstein[36] and Dolph[31] which prove that the absence of this interaction implies the existence (and uniqueness) of solution. Different proofs of these results, based on variational and fixed point techniques, are given.

Section 3 is devoted to the classical theorem by Ambrosetti and Prodi.[5] These authors studied the case in which $f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2$ (here, λ_1 and λ_2 denote, respectively, the first and second eigenvalue of the Laplace operator with zero Dirichlet boundary conditions). This condition means that the derivative of f jumps the first eigenvalue. Many improvements and related results appeared afterwards for the case in which the derivative jumps a finite number of eigenvalues of the spectrum. For this specific subject, we refer the reader to the papers quoted in Section 3. In contrast, in the rest of the paper, we restrict our attention to the less studied case of interaction with all but a finite number of eigenvalues. So, to conclude this section, we consider the case $f'(-\infty) < \lambda_1 < f'(+\infty) = +\infty$ and the result in Chang[19] and in De Figueiredo and Solimini[30] (see also Brézis and Nirenberg[17]) is given.

In the following sections the case $\lambda_1 < f'(-\infty) < f'(+\infty) = +\infty$ is treated either with Dirichlet boundary conditions (Section 4) or with Neumann boundary conditions (Section 5). This kind of asymmetric nonlinearities has been recently considered in Arcoya and Villegas,[10,11] Capietto and Dambrosio,[18] Dancer,[22] De Figueiredo,[25] De Figueiredo and Ruf,[28,29] Perera,[47] Pérez Sánchez,[48] Ruf and Srikanth[57] and Villegas.[59] Following the ideas of Arcoya and Villegas,[11] we use the Bifurcation Theory to study the Dirichlet problem in Section 4. However, for the Neumann problem in Section 5, variational techniques are applied as in Arcoya and Villegas[10] and Villegas.[59]

Notation. In the sequel we make use of the following notation:

Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$.

$|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$;

$L^p(\Omega)$, $1 \leq p \leq +\infty$ denote Lebesgue spaces; the norm in $L^p(\Omega)$ will be denoted by $\|.\|_p$;

$H^1(\Omega)$ denotes the usual Sobolev space endowed with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2\,dx$; $H^{-1}(\Omega)$ is the dual space of $H^1(\Omega)$.

$H_0^1(\Omega)$ denotes the subspace of the functions in $H^1(\Omega)$ which are zero (in the sense of the traces) in the boundary $\partial\Omega$ of Ω endowed with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx$;

$\rightharpoonup$ denotes weak convergence; $\longrightarrow$ denotes strong convergence;

$s^+ = \max\{s, 0\}$, $s^- = \min\{s, 0\}$;

$K, K_1, K_2, ..., C, C_1, C_2, ...$ denote (possibly different) positive constants;

$\lambda_1 < \lambda_2 \leq ... \leq \lambda_j \leq ...$ denote the eigenvalues of the Laplacian operator either with zero Dirichlet boundary conditions or zero Neumann boundary conditions. φ_j denotes the associated eigenfunction to λ_j.

2 No interaction with the spectrum

We consider in this section the Dirichlet b.v.p.

$$\left.\begin{array}{r} -\Delta u = f(x,u) + h(x), \quad x \in \Omega \\ u = 0, \ x \in \partial\Omega \end{array}\right\} \tag{2}$$

where $h \in L^2(\Omega)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies (1). Our goal consists in reminding the main existence results for the case in which the derivative of the nonlinearity does not interact with the spectrum of the Laplacian operator. In this way, our first result is the classical result of Hammerstein [36] (see also Iglisch [37]) about nonlinearities with derivative less than λ_1.

Theorem 1 (HAMMERSTEIN, 1930) a) *Suppose that $f'(-\infty), f'(+\infty) < \lambda_1$, then there exists at least one solution of (2) for every $h \in L^2(\Omega)$.*

b) *If, in addition, we assume that f is of class C^1 and $|f'(x,t)| \le m < \lambda_1$ for every $x \in \Omega$, $t \in \mathbb{R}$ then this solution is unique.*

Proof. The proof is based on variational methods. Let J be the functional defined in the Sobolev space $H_0^1(\Omega)$ by setting

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega hu\,dx, \tag{3}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. As it is known, J is of class C^1 and its critical points are just the (weak) solutions of (2) (see, for instance Ambrosetti, [3] De Figueiredo, [26] Rabinowitz [53]).

a) Since $f'(-\infty), f'(+\infty) < \lambda_1$, there exists $m < \lambda_1$ such that $|f(x,t)| \le m|t| + C$ and thus $|F(x,t)| \le mt^2/2 + C|t|$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Hence we deduce

$$J(u) \ge \frac{1}{2}\left(1 - \frac{m}{\lambda_1}\right)\|u\|^2 - C\|u\|.$$

This inequality implies that J is coercive. In addition, it is w.l.s.c. and therefore it attains its infimum at some $u \in H_0^1(\Omega)$ which necessarily is a critical point of J and so a solution of (2).

b) It suffices to observe that, in this case, $G(x,t) \equiv \lambda_1 t^2/2 - F(x,t)$ is convex in the variable t which allows us to conclude that

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right)\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega hu\,dx,$$

is a convex functional (as the sum of three convex functionals). □

The next result is due to Dolph. [31] The case in which $f'(-\infty)$ and $f'(+\infty)$ are between two consecutive eigenvalues of the Laplacian operator is considered.

Theorem 2 (DOLPH, 1949) a) *If* $\lambda_n < f'(-\infty), f'(+\infty) < \lambda_{n+1}$ *for some* $n \in \mathbb{N}$, *then there exists at least one solution of* (2) *for all* $h \in L^2(\Omega)$.

b) *If, in addition, we assume that* f *is of class* C^1 *and there exists* $\varepsilon > 0$ *such that* $\lambda_n + \varepsilon < f'(x,t) < \lambda_{n+1} - \varepsilon$ *for every* $x \in \Omega$ *and* $t \in \mathbb{R}$, *then this solution is unique.*

Proof. Let $\gamma = \frac{\lambda_n + \lambda_{n+1}}{2}$ and notice that (2) may be written as

$$u = K[N(u)], \ u \in L^2(\Omega),$$

where $K, N : L^2(\Omega) \longrightarrow L^2(\Omega)$ are defined by

$$Ku = v \text{ is the unique solution } v \text{ of } \left.\begin{array}{c} -\Delta v - \gamma v = u, \ x \in \Omega \\ v = 0, \ x \in \partial\Omega \end{array}\right\}$$

and

$$Nu(x) = f(x, u(x)) + h(x) - \gamma u(x), \ \forall x \in \Omega.$$

Hence, we have to prove the existence of a fixed point of $T = K \circ N$. This may be done by the Schauder fixed point theorem in case a) and by the Banach fixed point theorem in case b). We begin by the simpler case, i.e. case b):

b) Note that

$$\begin{aligned} \|Tu - Tv\|_2 &= \|K[N(u) - N(v)]\|_2 \\ &\leq \|K\|\|N(u) - N(v)\|_2 \\ &= \frac{2}{\lambda_{n+1} - \lambda_n}\|[f(u) - \gamma u] - [f(v) - \gamma v]\|_2 \\ &\leq \frac{2}{\lambda_{n+1} - \lambda_n}\left[\frac{\lambda_{n+1} - \lambda_n}{2} - \varepsilon\right]\|u - v\|_2, \end{aligned}$$

and thus T is a contraction which implies that (2) has a unique solution for all $h \in L^2(\Omega)$.

a) With respect to the proof of a), as mentioned above, it suffices to apply the Schauder theorem. In order to do this, observe that from the hypotheses on f, there exists a sublinear function g in $\overline{\Omega} \times \mathbb{R}$ such that we can write

$$f(x,t) = f'(+\infty)t^+ + f'(-\infty)t^- + g(x,t), \ \forall t \in \mathbb{R}$$

with $t^+ = \max\{t, 0\}$ and $t^- = \min\{t, 0\}$. Hence, there exists $\overline{\mu} \in (0, (\lambda_{n+1} - \lambda_n)/2)$ such that

$$|f(x,s) - \gamma s| \leq \overline{\mu}|s|, \ \forall s \in \mathbb{R}$$

and thus

$$\begin{aligned}\|Tu\|_2 &\leq \|K\|\|N(u)\|_2 \\ &\leq \frac{2}{\lambda_{n+1}-\lambda_n}\left[\|f(x,u)-\gamma u\|_2+\|h\|_2\right] \\ &\leq \frac{2}{\lambda_{n+1}-\lambda_n}\left[\overline{\mu}\|u\|_2+\|h\|_2\right] \\ &\leq \mu\|u\|_2+C,\end{aligned}$$

with $0 < \mu < 1$. So choosing $r > 0$ such that $\mu r + C < r$, we deduce that $\|Tu\|_2 \leq r \ \forall \|u\|_2 \leq r$, i.e. $T(\overline{B}(0,r)) \subset \overline{B}(0,r)$. Finally, the compactness of K and the continuity of the Nemistky operator N implies the compactness of T and so we can use the Schauder theorem to conclude the proof of a). □

A variational proof of a). We can apply the Rabinowitz [52] saddle point Theorem to obtain a different proof of part a) of the theorem above. Indeed, we consider again the C^1−functional J given by (3). Taking

$$\lambda_n < \mu < f'(-\infty), f'(+\infty) < \overline{\mu} < \lambda_{n+1},$$

it is easy to verify that

$$\frac{1}{2}\mu t^2 - C_1 \leq F(x,t) \leq \frac{1}{2}\overline{\mu} t^2 + C_2, \ \forall t \in \mathbb{R}, \forall x \in \Omega$$

and thus

$$\frac{1}{2}\|u\|^2 - \frac{\overline{\mu}}{2}\|u\|_2^2 - C_2|\Omega| - \|h\|_2\|u\|_2 \leq J(u) \leq \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|_2^2 + C_1|\Omega| + \|h\|_2\|u\|_2.$$

Splitting $H_0^1(\Omega)$ into $H_0^1(\Omega) = V \oplus V^\perp$ with $V = \langle \varphi_1, \varphi_2, \dots, \varphi_n \rangle$, we deduce from the variational characterization of the eigenvalues λ_j that

$$J(u) \leq \frac{1}{2}\left(1 - \frac{\mu}{\lambda_n}\right)\|u\|^2 + C_1|\Omega| + \frac{\|h\|_2}{\lambda_1}\|u\|, \ \forall u \in V$$

and

$$J(u) \geq \frac{1}{2}\left(1 - \frac{\overline{\mu}}{\lambda_{n+1}}\right)\|u\|^2 - C_2|\Omega| - \frac{\|h\|_2}{\lambda_1}\|u\|, \ \forall u \in V^\perp.$$

So, it is possible to choose $R > 0$ such that

$$\max_{u \in V, \|u\| = R} J(u) < \inf_{u \in V^\perp} J(u).$$

On the other hand, J satisfies the Palais-Smale condition, i.e. *if* $\{u_n\} \subset H_0^1(\Omega)$ *is such that* $\{J(u_n)\}$ *is bounded and* $\{J'(u_n)\}$ *tends to zero in the dual space of* $H_0^1(\Omega)$ *then* $\{u_n\}$ *has a convergent subsequence.* Indeed, it suffices to prove that all sequences satisfying the above hypotheses are bounded in $H_0^1(\Omega)$. Assume, by contradiction, that $\|u_n\| \to +\infty$ and observe that using $\lim_{n\to+\infty} J'(u_n)(\varphi)/\|u_n\| = 0$ and taking $v_n \equiv u_n/\|u_n\|$, we obtain

$$\lim_{n\to+\infty} \int_\Omega \nabla v_n \cdot \nabla\varphi\, dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\varphi\, dx - \int_\Omega \frac{h\varphi}{\|u_n\|}\, dx = 0,$$

for every $\varphi \in H_0^1(\Omega)$. Passing to a subsequence, if necessary, we may assume without loss of generality that $\{v_n\} \rightharpoonup v$ in $H_0^1(\Omega)$, $\{v_n\} \to v$ in $L^2(\Omega)$, $\{v_n(x)\} \to v(x)$ a.e. $x \in \Omega$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\varphi\, dx = \int_\Omega \left(f'(+\infty)v^+ + f'(-\infty)v^-\right)\varphi\, dx$$

and hence

$$\int_\Omega \nabla v \cdot \nabla\varphi\, dx = \int_\Omega \left(f'(+\infty)v^+ + f'(-\infty)v^-\right)\varphi\, dx,$$

i.e. v is a solution of the problem

$$\left.\begin{array}{c} -\Delta v = \alpha v^+ + \beta v^-, \quad x \in \Omega \\ v = 0, \quad x \in \partial\Omega \end{array}\right\} \tag{4}$$

with $\alpha = f'(+\infty)$ and $\beta = f'(-\infty)$. This implies that $v = 0$ and this is a contradiction because we deduce that

$$0 = \lim_{n\to+\infty} J'(u_n)(v_n) = 1 - \lim_{n\to+\infty} \int_\Omega f(x,u_n)v_n\, dx - \int_\Omega hv_n\, dx = 1.$$

Therefore, $\{u_n\}$ is bounded and the Palais-Smale condition has been verified. All hypotheses of the Rabinowitz theorem are satisfied which proves part a) of the Theorem. □

Remark 3 : Fučik spectrum. In the above variational proof or, more specifically, in the verification of the Palais-Smale condition, it is essential the fact that problem (4) has zero as a unique solution if α and β lie between two consecutive eigenvalues. The set Σ of the pairs (α, β) such that (4) has nontrivial solutions is called the Fučik spectrum. It was Fučik [33] who gave a complete description of it in the case $N = 1$. The description in the case $N \geq 2$ is still an open problem. Some interesting results have been given in Arias and Campos,[12,13] Dancer,[24] De Figueiredo and Gossez,[27] De Figueiredo and Ruf[29] and Gallouet and Kavian.[34] Indeed, in Dancer,[24] it is shown that the two lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in Σ. In Gallouet and Kavian[34] it is proved that from each (λ_k, λ_k) emanates a curve in Σ. A variational characterization of the curve S_1 emanating from (λ_2, λ_2) is deduced in De Figueiredo and Gossez.[27] In addition, in this paper the authors proved that S_1 is asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$. This is in contrast with the case of Neumann boundary conditions which is also studied in the quoted paper. In fact, for the Neumann b.v.p. the asymptotic behavior of the curve emanating from (λ_2, λ_2) is different if $N \geq 2$ (asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$) that is if $N = 1$ (asymptotic to $\{\pi^2/4\} \times \mathbb{R}$ and $\mathbb{R} \times \{\pi^2/4\}$ if $\Omega = (0,1)$). In Arias and Campos[12] the description of the radial spectrum is given.

3 Ambrosetti-Prodi problem and related results

The previous results have a common feature: *the derivative of the nonlinearity does not interact with the spectrum of the Laplacian operator.* However, there are a lot of cases where this interaction appears. For instance, we can consider the study of the existence of positive solutions for the b.v.p.

$$\left.\begin{array}{c} -\Delta u = f(x, u), \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega \end{array}\right\} \tag{5}$$

where $f \in C(\overline{\Omega} \times \mathbb{R}_0^+, \mathbb{R})$ satisfies $f(x, 0) \geq 0$, for every $x \in \Omega$. As it is well-known (see Ambrosetti and Hess[4]), one way to study this problem consists in extending the nonlinearity to $\overline{\Omega} \times \mathbb{R}$ by taking $f(x, u) = f(x, 0)$ for $x \in \overline{\Omega}$, $u < 0$. Indeed, in this way, by the maximum principle, the solutions of the extended problem are just the positive solutions of our problem. Observe that we then have $f'(-\infty) = \lim_{u \to -\infty} f(x, u)/u = 0$. Therefore, in all cases in which $f'(+\infty) > \lambda_1$, we have an asymmetric problem with interaction of the derivative of the (extended) nonlinearity with the spectrum.

In the paper by A. Ambrosetti and G. Prodi[5], the case in which the derivative of the nonlinearity jumps the first eigenvalue of the Laplacian operator was studied for the first time. Specifically, they proved

Theorem 4 (Ambrosetti and Prodi, 1973) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,\alpha}$ $(0 < \alpha < 1)$ and $f \in C^2(\mathbb{R})$ satisfying*

i) $f(0) = 0$,

ii) $f''(t) > 0$ *for every* $t \in \mathbb{R}$,

iii) $\lim_{t\to-\infty} f'(t) = f'(-\infty) \in (0, \lambda_1)$,

iv) $\lim_{t\to+\infty} f'(t) = f'(+\infty) \in (\lambda_1, \lambda_2)$.

Then there exists in $C^{0,\alpha}(\overline{\Omega})$ a closed connected C^1−manifold $\mathcal{M}$ of codimension 1, such that $C^{0,\alpha}(\overline{\Omega}) - \mathcal{M}$ consists exactly of two connected components A_1, A_2 with the following properties:

a) *If $h \in A_1$ then problem* (2) *has no solution in $C_0^{2,\alpha}(\overline{\Omega})$.*

b) *If $h \in A_2$ then problem* (2) *has exactly two solutions in $C_0^{2,\alpha}(\overline{\Omega})$.*

c) *If $h \in \mathcal{M}$ then problem* (2) *possesses a unique solution in $C_0^{2,\alpha}(\overline{\Omega})$.* □

The proof of this theorem is based on theorems of global inversion of mappings with singularities. This can be seen in the original paper or also in the book of Ambrosetti and Prodi. [6] We remark explicitly that the previous result gives us the exact number of solutions of the b.v.p. (2). Since this paper, the problem of studying the existence of solutions, or more specifically, the lower bounds of the number of solutions, of b.v.p. with nonlinearities jumping the first eigenvalue is called as Ambrosetti-Prodi problem. The literature is very vaste. Among many others, we cite Amann and Hess, [2] Berestycki, [15] Berger and Podolak, [16] Chang, [19] Dancer, [24] De Figueiredo and Solimini, [30] Fučik, [33] Gallouet and Kavian, [34] Kazdan and Warner, [40] Podolak [49] and Ruf [55] for the case of a nonlinearity with $f'(-\infty) < \lambda_1 < f'(+\infty)$.

The more general case in which the nonlinearity interacts with a finite number of eigenvalues (without necessarily including λ_1) has been extensively studied by Arias and Campos, [14] Costa, [20] Costa,De Figueiredo and Srikanth, [21] Domingos and Ramos, [32] Lazer and McKenna, [42,43,44] Micheletti and Pistoia, [45] Orsina, [46] Ramos [54] and Ruf and Solimini. [56] In order to have a reasonable length for these notes we do not include here the results in this field. We only refer the reader to the quoted papers and the references therein.

In the following, we study cases in which the interaction is with infinitely many eigenvalues. We begin by studying the case $f'(-\infty) < \lambda_1 < f'(+\infty) = +\infty$. Observe that the meaning of this condition is that the derivative crosses all the spectrum. Actually we can state (see De Figueiredo [26]) the existence of at least one or two solutions without assuming $f'(+\infty) = +\infty$. However, by definiteness we impose that this equality holds.

Theorem 5 *Let $h \in C^{\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ be a function such that $\int_\Omega h\varphi_1\,dx = 0$. Assume that $-\infty < f'(-\infty) < \lambda_1$ and*

i) *If* $N \geq 2$*, there exist* $\sigma \in (1, 2^*)$ *and* $K_1, K_2 > 0$ *constants such that*

$$|f(x,s)| \leq K_1 + K_2|s|^\sigma\,, \quad \forall x \in \overline{\Omega}\,, \quad \forall s \in \mathbb{R}$$

(*where* $2^* = \frac{N+2}{N-2}$ *for* $N \geq 3$*, and* $2^* = +\infty$ *for* $N = 2$).

ii) *There exist* $s_0 > 0$ *and* $\theta \in (0, \frac{1}{2})$ *such that*

$$0 < F(x,s) \leq \theta s f(x,s)\,, \quad \forall x \in \overline{\Omega}\,, \; \forall s \geq s_0$$

where $F(x,s) = \int_0^s f(x,t)\,dt$ *is a primitive of* f.

Then there exists $t_0 \in \mathbb{R}$ *such that the problem*

$$\left.\begin{array}{c} -\Delta u = f(x,u) + t\varphi_1 + h(x), \quad x \in \Omega \\ u = 0, \; x \in \partial\Omega \end{array}\right\} \qquad (P_t)$$

has

a) *no solution if* $t > t_0$,

b) *at least one solution if* $t = t_0$*, and*

c) *at least two solutions if* $t < t_0$.

Remark 6 Hypothesis ii) is a condition of "superlinearity" on f which may be found in Ambrosetti and Rabinowitz.[7] It may be seen (for instance, in De Figueiredo[26]) that this implies the existence of $K > 0$ such that

$$f(x,s) \geq K s^{\frac{1}{\theta}-1}, \; \forall s \geq s_0 \qquad (6)$$

and so $f'(+\infty) = +\infty$.

Sketch of the proof. We follow closely the idea in Chang[19] and in De Figueiredo and Solimini.[30] Take $f'(-\infty) < \mu < \lambda_1$. Using that $f(x,s) > \mu s - C$ for every $x \in \Omega$ and $s \in \mathbb{R}$, we deduce that, since $\mu < \lambda_1$, the unique solution χ_t of the problem

$$\left.\begin{array}{c} -\Delta u = \mu u - C + t\varphi_1 + h(x), \quad x \in \Omega \\ u = 0, \; x \in \partial\Omega \end{array}\right\}$$

is a lower solution of (P_t). In addition by the maximum principle it is easy to prove that any upper solution is greater than χ_t. Therefore, to prove the existence of, at least, one solution it suffices to show the existence of an upper solution. Now, by the arguments in Kannan and Ortega[39] and in Kazdan and Warner,[40] we can prove that for $t << 0$ there exists an upper solution. Hence, the set Γ of $t \in \mathbb{R}$ for which there exists an upper solution is not empty. Using that every upper solution of (P_s) is an strict upper solution of (P_t) for all $t < s$, it is easy to show that Γ is a closed interval unbounded from below and bounded from above. Take $t_0 = \max \Gamma$. Parts a) and b) of the theorem are proved. For $t < t_0$, we have a pair of ordered strict lower and upper solutions, (χ_t, u) with u an upper solution of (P_{t_0}). This implies by the arguments in Brézis and Nirenberg [17] that there exists a solution u_1 which corresponds to a local minimum of the functional J defined in $H_0^1(\Omega)$ by setting

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(x,u) \, dx - t \int_\Omega \varphi_1 u \, dx - \int_\Omega h \, u \, dx, \quad u \in H_0^1(\Omega).$$

In addition, since $f'(+\infty) = +\infty$ (it suffices that $f'(+\infty) > \lambda_1$), we have $\lim_{s \to +\infty} J(s\varphi_1) = -\infty$ and consequently there are $r > 0$ and $e \in H_0^1(\Omega)$ such that $J(e) < J(u_1)$ (with u_1 a local minimum of J). Therefore the geometrical hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz [7] are satisfied. To conclude the proof of part c) we just have to show that J satisfies the Palais-Smale condition. But, observing that from the hypotheses on f the following inequality is deduced.

$$\frac{1}{\theta} F(x,s) \leq s f(x,s) + C\lambda_1 s^2, \ \forall x \in \Omega, \ \forall |s| \geq R$$

with $C \in (0, \frac{1}{2} - \theta)$ and $R >> 0$, and by using the classical ideas of Ambrosetti and Rabinowitz [7] to prove the Palais-Smale condition for superquadratic functionals as J, it is then easily verified this compactness condition for J (see Lemma 3.2 in Arcoya and Boccardo [8] for the details). □

Remark 7 At the beginning of this section, we pointed out that the study of the existence of positive solutions for b.v.p.'s with a superlinear (at $+\infty$) nonlinearity can be seen as an asymmetric problem. In addition to embed this kind of problem into the more general framework of the asymmetric b.v.p., this point of view allows also to revise their existence results as particular cases. In this way, using the previous theorem we can easily deduce the classical Ambrosetti and Rabinowitz [7] Theorem for the existence of positive solutions of (5), i.e. that if we assume i), ii) and that $\lim_{s \to 0^+} f(x,s)/s = 0$ (uniformly in $x \in \Omega$ then there exists a positive solution of (5). Indeed, taking $h \equiv 0$ in Theorem 5 and since $u = 0$ is a trivial solution of the problem, we get that, in this case, $0 \in \Gamma$ and so $t_0 \geq 0$ which implies the assertion by c) of the quoted theorem.

4 Interaction with all but a finite number of eigenvalues

This section is devoted to study the existence of solutions of Dirichlet problems like

$$\left.\begin{array}{rcl} -\Delta u &=& f(x,u), \ \ x\in\Omega \\ u &=& 0, \ \ x\in\partial\Omega \end{array}\right\} \tag{7}$$

where $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function satisfying $f'(-\infty)=\lambda\in\mathbb{R}$ and $f'(+\infty)=+\infty$. Previous results have been given in Dancer,[22] De Figueiredo,[25] Ruf and Srikanth[57] (see also Section 6 in Dancer[23]). In these papers, the authors study the case of nonlinearities f which are superlinear at $+\infty$ and, on the other hand, at $-\infty$ satisfy

(f_1) There exists $\lambda\in\mathbb{R}$ such that

$$\lim_{s\to-\infty}\frac{f(x,s)}{s}=\lambda, \text{ uniformly in } x\in\overline{\Omega}.$$

The results in Dancer[22] and Ruf and Srikanth[57] are for Dirichlet problems like (7) with $N=1$. The case of a general domain $\Omega\subset\mathbb{R}^N$ ($N\geq 1$) is considered in De Figueiredo[25] where the author assumes a very technical set of assumptions for the nonlinearity. More recently, Capietto and Dambrosio[18] studied the case $N=1$ by using a continuation theorem and a time-map technique based on a generalized Fučik spectrum.

Here, our idea is to apply a bifurcation theory on the parameter λ and, for a suitable truncated problem of (7), to prove that condition (f_1) implies the existence of a branch of solutions bifurcating from infinity at λ_1. The behavior of this branch may be studied if, in addition, we assume that $f(x,0)>0$ for every $x\in\overline{\Omega}$ and (7) admits a positive supersolution. In this case, it is standard to deduce by iterative methods the existence of a positive solution of (7). (Observe that the asymmetric hypotheses on f are not essential in order to find this solution.) However, we show that the branch does not contain positive solutions. This allows to obtain sufficient conditions in the case $\lambda\geq\lambda_1$ for the existence solutions which are negative in some subset of Ω.

For this purpose, assume that $f(x,s)$ is a locally Lipschitz function in $\overline{\Omega}\times\mathbb{R}$ satisfying (f_1) and that $u_0\in C^2(\overline{\Omega})$ is a supersolution of (7). Let $M=\max_{x\in\overline{\Omega}}u_0(x)$ and consider the truncation $f_0(x,s)$ of $f(x,s)$ given by

$$f_0(x,s)=\begin{cases} f(x,s), & \text{if } s\leq M, \ x\in\overline{\Omega} \\ f(x,M), & \text{if } s>M, \ x\in\overline{\Omega}. \end{cases}$$