Roger Moser

PARTIAL REGULARITY *for* HARMONIC MAPS RELATED PROBLEMS

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by

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Preface

This monograph contains material from several research papers and lectures I gave in Bonn, Leipzig, New York, and Fribourg on various occasions, all of them about different aspects of the same problem. In an attempt to make the work nearly self-contained, I also included many additional paragraphs and most of the proofs of the auxiliary results. It is assumed, however, that the reader is familiar with the basic theory of linear elliptic and parabolic partial differential equations, and with the elementary notions of Riemannian geometry.

The aim of the book is to explain the methods that have been developed in the last decades to prove partial regularity for harmonic maps, and also to show how these methods can be extended to related problems. This includes perturbations of the harmonic map problem as well as associated parabolic problems. Both types may be of interest in applications from physics or possibly other sciences.

Roger Moser
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Chapter 1

Introduction

Variational principles play an important role in both geometry and physics, and one of the key problems with applications in both fields is the variational problem associated to the Dirichlet energy of maps between Riemannian manifolds. The critical points of this functional are called harmonic maps.

Harmonic maps have successfully been used in many instances in order to understand the geometry of the involved manifolds. The Dirichlet energy also appears in the context of various theories from physics; for instance, in connection with liquid crystals, ferromagnetic materials, or superconductors. Another motivation to study harmonic maps is the fact that many principles that hold for the Dirichlet energy and their critical points, the harmonic maps, have counterparts in other theories. Methods that have been developed for harmonic maps can sometimes also be applied to other problems, and vice versa. For example, the problems of minimal surfaces or Yang-Mills fields show a similar behavior as the harmonic map problem in certain aspects.

The Dirichlet energy is one of the simplest possible functionals involving first derivatives. In the theory of harmonic maps, however, it is studied on a non-linear space, which gives rise to non-linear Euler-Lagrange equations for its critical points. These non-linearities make it a challenge to study the solutions. Whereas it is not difficult to construct weak solutions in general—for instance by the direct method from the calculus of variations—, in order to see how regular the weak solutions are, one needs quite subtle arguments. The methods that are used here take advantage of the special structure of the equations, and involve some rather deep results from harmonic analysis. It is the aim of this monograph to explain these arguments and to show how they can be used to prove regularity, or partial regularity (for the solutions are not regular everywhere in general) of solutions of related problems, including perturbations and generalizations of the harmonic map problem as well as parabolic problems associated to the Dirichlet energy.

1.1 Harmonic Maps

Suppose M and N are smooth Riemannian manifolds of dimensions m and k, respectively. We assume throughout the book that N is compact and has no boundary. The manifold M, in contrast, may be non-compact and may have a boundary. We consider maps $u : M \to N$, therefore we call M the domain and N the target manifold. We now give a definition of the Dirichlet energy of a sufficiently smooth map $u : M \to N$ (for instance, $u \in C^1(M; N)$). Because regularity is a local problem, and because the geometry of M is not so important for the arguments, we will assume later that M is an open subset of the Euclidean space $\mathbb{R}^m$ (and we will then write Ω instead of M). For completeness, however, we at least write down the problem for a general manifold M here.

A map $u : M \to N$ induces a metric on the vector bundle $T^*M \otimes u^{-1}TN$ over M (with fiber $T^*_xM \otimes T_{u(x)}N$ at the point $x \in M$). We write $\langle \cdot, \cdot \rangle_{T^*M \otimes u^{-1}TN}$ for this metric. If u is continuously differentiable on M, then the first derivative du is a continuous section of this vector bundle. Hence the function

$$e(u) = \frac{1}{2}|du|^2_{T^*M \otimes u^{-1}TN} = \frac{1}{2}\langle du, du \rangle_{T^*M \otimes u^{-1}TN}$$

is well-defined. We set

$$E(u) = \int_M e(u)\, d\mu_M, \tag{1.1}$$

where μ_M is the measure induced by the Riemannian metric on M, provided that the integral converges. This number is called the Dirichlet energy of the mapping u.

Next we give this expression in local coordinates. Suppose we work in a coordinate chart M' of M, and the metric is given there by the matrix $\gamma = (\gamma_{\alpha\beta})_{1 \le \alpha, \beta \le m}$ in local coordinates. Suppose further that the image of M' under u is contained in a coordinate chart of N. The metric of N is given by $g = (g_{ij})_{1 \le i,j \le k}$ in the corresponding local coordinates. We write $(\gamma^{\alpha\beta}) = \gamma^{-1}$, thus the metric of T^*M is given by this matrix. Then we have

$$e(u) = \frac{1}{2} \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{k} \gamma^{\alpha\beta} g_{ij} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta},$$

and the Dirichlet integral is given locally by the integral

$$E(u; M') = \frac{1}{2} \sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{k} \int_{M'} \gamma^{\alpha\beta} g_{ij} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \sqrt{|\det \gamma|}\, dx.$$

When we compute the Euler-Lagrange equations for critical points of this functional, we find

$$\sum_{\alpha,\beta=1}^{m} \gamma^{\alpha\beta} \left(\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - \sum_{\delta=1}^{m} \Gamma^{\delta}_{\alpha\beta} \frac{\partial u^i}{\partial x^\delta} + \sum_{j,l=1}^{k} C^i_{jl} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^l}{\partial x^\beta} \right) = 0$$

for $i = 1, \ldots, k$. Here $\Gamma^{\delta}_{\alpha\beta}$ are the Christoffel symbols in M', and C^i_{jl} the Christoffel symbols for the local coordinates on N. If D denotes the Levi-Civita connection on $T^*M \otimes u^{-1}TN$, a coordinate free representation of these equations is

$$\operatorname{trace}(Ddu) = 0.$$

The point of view we have taken so far is not optimal when we consider the variational problem associated to the Dirichlet energy and weak solutions of the equations. Normally, it is natural to study such a functional in a Sobolev space; in this particular case, a space of maps with square integrable first derivatives would be an obvious choice. Unfortunately, maps in such a Sobolev space are not continuous in general, and the assumption that the image of M' under u is in a coordinate chart of N is not justified, even if M' is very small. In other words, we cannot carry out the same calculations as above for maps in a Sobolev space.

The easiest way to overcome this difficulty is to assume that N is isometrically embedded in a Euclidean space $\mathbb{R}^n$. This appears to be a restriction at a first glance, but in fact such an isometric embedding always exists, owing to the so-called Nash-Moser embedding theorem which was proved by J. Nash [1954; 1956] (and the methods were improved by J. Moser [1961]). We regard N as a submanifold of $\mathbb{R}^n$ in the rest of this book. We can then consider the Sobolev space

$$H^1(M; N) = \left\{ u \in H^1(M; \mathbb{R}^n) : \ u(x) \in N \text{ for almost every } x \in M \right\},$$

where $H^1(M;\mathbb{R}^n)$ is the Sobolev space of all weakly differentiable maps $u: M \to \mathbb{R}^n$ with

$$E(u) = \frac{1}{2}\int_M |du|^2_{T^*M\otimes\mathbb{R}^n}\, d\mu_M < \infty. \tag{1.2}$$

If u is in the space $H^1(M;N)$, then $du(x)$ is in $T_x^*M \otimes T_{u(x)}N$ at almost every point $x \in M$. Hence the number $E(u)$ defined by (1.2) coincides with the Dirichlet energy given by (1.1) whenever the latter is well-defined.

The new representation of the energy functional gives rise to a new version of the Euler-Lagrange equations. Suppose the second fundamental form of the submanifold N is denoted by A. Then it can be shown that critical points of E satisfy

$$\Delta^M u + \sum_{\alpha,\beta=1}^{m} \gamma^{\alpha\beta} A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right) = 0 \quad \text{on } M,$$

where $\Delta^M = \operatorname{div}^M(\nabla^M u)$ is the Laplace-Beltrami operator on M (and ∇^M denotes the gradient on M).

We assume henceforth that Ω is an open domain in $\mathbb{R}^m$, and we replace M by Ω. This simplification will spare us some technical work while preserving all the important ideas behind the arguments that are presented in the subsequent chapters. We then also use a simplified notation. The gradient on Ω is denoted by ∇, the Laplacian by Δ. We consider the Dirichlet energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{m\times n} = \mathbb{R}^m \otimes \mathbb{R}^n$. The Euler-Lagrange equation for critical points of E becomes

$$\Delta u + A(u)(\nabla u, \nabla u) = 0 \quad \text{in } \Omega, \tag{1.3}$$

where we use the abbreviation

$$A(u)(\nabla u, \nabla u) = \sum_{\alpha=1}^{m} A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\alpha}\right).$$

We will show in Section 3.2 how this equation is derived. (For a general domain manifold M, the computations are essentially the same.) It will also be shown that (1.3) is equivalent to

$$\Delta u(x) \perp T_{u(x)}N$$

for almost every $x \in \Omega$.

A map $u \in H^1(\Omega; N)$ which satisfies (1.3) in the weak sense is called a weakly harmonic map. The regularity of weakly harmonic maps is the first problem we study in this book. It turns out that weakly harmonic maps may have singularities in general. For instance, the map

$$u(x) = \frac{x}{|x|},$$

mapping $\mathbb{R}^m \backslash \{0\}$ onto the unit sphere

$$\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m : \ |x| = 1\},$$

satisfies the equation

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \mathbb{R}^m \backslash \{0\}. \tag{1.4}$$

This, however, is (1.3) in the case $N = \mathbb{S}^{m-1}$ (as will be shown in Section 3.5). If $m \geq 3$, the map u satisfies (1.3) in the weak sense in $\mathbb{R}^m$. If it is restricted to a bounded domain $\Omega \subset \mathbb{R}^m$, it has finite energy and is thus a weakly harmonic map. But obviously u has a singularity at 0. The situation is even worse: There exists an example, constructed by T. Rivière [1995], of a weakly harmonic map that is discontinuous everywhere in its domain.

Fortunately, this is not the end of the story. On the one hand, the dimension of the domain plays a role here. For $m = 1$, we have an ordinary differential equation (and its solutions are geodesics). Regularity is not an issue here, and we don't even consider this case any more henceforth. The case of two-dimensional domains is special, too. We will see that for $m = 2$, every weakly harmonic map is smooth. The example of T. Rivière is in three dimensions, and from it, non-regular harmonic maps in higher dimensions can also be constructed. But despite these discouraging counter-examples, all is not lost even if $m \geq 3$. When we go back to the variational problem for the Dirichlet energy E, we find that it is natural to impose an additional condition, and then certain partial regularity results can be proved. Equation (1.3) is equivalent to the condition

$$\left.\frac{d}{ds}\right|_{s=0} E(\pi_N \circ (u + s\phi)) = 0$$

for any smooth map $\phi : \Omega \to \mathbb{R}^n$ with compact support, where π_N is the so-called nearest point projection onto N, i. e., the unique map that assigns to every point y in a neighborhood of N the point on N that minimizes the distance to y. The family of maps $\pi_N \circ (u + s\phi)$ is a natural variation of u in $H^1(\Omega; N)$, but this is not the only kind of variation we can consider. If we have a smooth map $\psi : \Omega \to \mathbb{R}^m$ with compact support, then the family given by $u_s(x) = u(x + s\psi(x))$ gives another variation of u. A weakly harmonic map $u \in H^1(\Omega; N)$ is called stationary if

$$\left.\frac{d}{ds}\right|_{s=0} E(u_s) = 0 \tag{1.5}$$

for all such variations. A smooth harmonic map is automatically stationary, but a weakly harmonic map not necessarily.

Stationary weakly harmonic maps can still have singularities. In fact the maps $u(x) = x/|x|$ discussed above are stationary for each $m \geq 3$. But in contrast to weakly harmonic maps in general, the singular set, i. e., the minimal relatively closed set in Ω such that u is smooth on the complement, is always small if the maps are stationary. For $u(x) = x/|x|$, the singular set consists of one single point. In general one can prove that a stationary weakly harmonic map is smooth away from a singular set of vanishing $(m-2)$-dimensional Hausdorff measure.

What makes stationary weakly harmonic maps special? When we want to prove regularity, the crucial observation is that the decay of the Dirichlet energy, restricted to concentric balls of shrinking radii, is described by a very convenient formula if we have a stationary weakly harmonic map. As a consequence, we have a good estimate for the mean oscillation (see the definition in Section 2.3) of the map. The problem of proving partial regularity of the kind as described above is then reduced to showing that a weakly harmonic map of small mean oscillation is always smooth. To do that, one can work with equation (1.3) again.

We now try to give a glimpse of how (1.3) can be exploited, with arguments based on ideas of F. Hélein [1990; 1991a; 1991b], in order to prove regularity. These arguments and the ones mentioned previously are given in detail in Chapter 3; here we just try to give an intuitive idea of how the structure of the equation comes into play. We consider again the special case where N is a sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then equation (1.3) can be written in the form (1.4), but there exists yet another equivalent system of equations: A map $u = (u^1, \ldots, u^n) \in H^1(\Omega; \mathbb{S}^{n-1})$ satisfies (1.4) if and only if

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0 \quad \text{in } \Omega \tag{1.6}$$

for all $i, j = 1, \ldots, n$. Using the exterior product $\wedge : \mathbb{R}^n \times \mathbb{R}^n \to \Lambda_2\mathbb{R}^n$, we can also write

$$\operatorname{div}(u \wedge \nabla u) = 0 \quad \text{in } \Omega$$

for (1.6). If we already know that u is regular, the equivalence of (1.4) and (1.6) can be checked directly, using the properties of the exterior product. It is shown in Section 3.5 that they are equivalent whenever $u \in H^1(\Omega; \mathbb{S}^{n-1})$. We have

$$|\nabla u|^2 = |u \wedge \nabla u|^2 = \sum_{i,j=1}^{n} u^i \nabla u^j \cdot (u^i \nabla u^j - u^j \nabla u^i),$$

because $u \perp \frac{\partial u}{\partial x^\alpha}$ almost everywhere for every $\alpha = 1, \ldots, m$. Thus the energy density

$$e(u) = \frac{1}{2}|\nabla u|^2$$

can be written in a very special form: the sum of products of

- a gradient,
- a divergence free vector field, and
- a function with a mean oscillation for which one has good estimates.

Then we need a few sophisticated tool from harmonic analysis. We will discuss them in the next section, along with some other analytic preliminaries. Here we just describe them briefly: A compensated compactness principle due to R. Coifman, P. L. Lions, Y. Meyer, and S. Semmes [1993] gives an estimate of the product of a gradient and a divergence free vector field in a Hardy space, in terms of the L^2-norms of the factors. This Hardy space is the dual to the space of functions of bounded mean oscillation, according to a result by C. Fefferman and E. M. Stein [1972]. In particular the product above has an estimate that allows to draw the following conclusion: If the energy of u is sufficiently small in a given ball, then the decay of the energy in concentric balls of shrinking radii is even better than the decay that condition (1.5) implies. In fact it is good enough to apply a well-known decay lemma of C. B. Morrey and conclude that u is Hölder continuous at least near the center of the ball. Continuity is the first step on the way to prove regularity, and it turns out that it is the most difficult. Once we know u is continuous, higher regularity is proved relatively easily.

For other target manifolds, this method does no longer work in the same way, because there is no representation of the harmonic map equation like

(1.6) in general. But with the help of additional arguments, due essentially to F. Hélein [1991a], similar arguments can still be applied.

1.2 Related Elliptic Equations

In applications, the harmonic map equation is often encountered not in its "pure" form, but with additional terms. For instance, the functional we study may be the Dirichlet energy plus some other, lower order terms. We consider briefly one such example as a motivation for the problems discussed in Chapter 4, namely an energy functional from the theory of micromagnetics. Many other functionals from physics or geometry could be considered instead, but we use this one because it is also relevant for another problem we will study later.

We assume $m = 3$. The bounded domain Ω represents the shape of a ferromagnetic sample and $u : \Omega \to \mathbb{R}^3$ its magnetization vector field. At low temperatures, one can assume that the magnitude of u is constant; for an appropriate choice of units this means $|u| = 1$ almost everywhere. That is, we have the target manifold $N = \mathbb{S}^2$. The energy assigned to u consists of several components:

- The highest order contribution is the Dirichlet energy, multiplied by a material constant d^2. That is, we have the term

$$\frac{d^2}{2} \int_\Omega |\nabla u|^2 \, dx.$$

  In this context, this is also called the exchange energy. The constant d is called the exchange length. The exchange energy penalizes spatial variations of the magnetization vector field, which reflects the fact that neighboring magnetization vectors have a tendency for parallel alignment.
- Crystalline anisotropies in the material can be modeled by a term of the form

$$\int_\Omega \Phi(u) \, dx$$

  for a function $\Phi : \mathbb{S}^2 \to \mathbb{R}$. We assume Φ is smooth.
- The magnetization induces a magnetic stray field $h : \mathbb{R}^3 \to \mathbb{R}^3$ which has a certain energy. According to the static Maxwell equa-

tions, the stray field satisfies

$$\begin{aligned}\operatorname{curl} h &= 0 \quad \text{in } \mathbb{R}^3,\\ \operatorname{div}(u+h) &= 0 \quad \text{in } \mathbb{R}^3,\end{aligned}$$

where u is extended by 0 outside of Ω. The first equation implies that $-h$ can be written as the gradient of a function U. By the second equation, this U is a solution of

$$\Delta U = \operatorname{div} u \quad \text{in } \mathbb{R}^3$$

in the distribution sense. There exists exactly one solution such that the integral

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2 \, dx = \frac{1}{2}\int_\Omega u \cdot \nabla U \, dx$$

is finite, and for this choice of U this integral gives the third contribution to the micromagnetic energy. It is called the magnetostatic energy.

- If there is an external field $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we also have the energy

$$-\int_\Omega u \cdot H \, dx.$$

The full micromagnetic energy is the sum of all four terms, that is, the functional

$$F(u) = \int_\Omega \left(\frac{d^2}{2}|\nabla u|^2 + \Phi(u) + \frac{1}{2} u \cdot \nabla U - u \cdot H \right) dx.$$

A discussion of the physical background of this energy can be found, e. g., in the books by A. Aharoni [2001] or by A. Hubert and R. Schäfer [1998].

Critical points of the functional F satisfy

$$d^2(\Delta u + |\nabla u|^2 u) - \operatorname{grad} \Phi(u) + (H - \nabla U)^\top = 0 \quad \text{in } \Omega,$$

where grad denotes the gradient on $\mathbb{S}^2$ and $(\,\cdot\,)^\top$ denotes the orthogonal projection $X^\top = X - \langle u, X\rangle\, u$ onto the tangent space of $\mathbb{S}^2$ at the image point of u. The first two terms are (up to the factor d^2) the same as in (1.4). They are the highest order contribution, hence they should be the most important terms when we study the regularity of solutions. It turns out that this is true. But when we try to prove regularity with the methods for harmonic maps, we have to handle quite subtle tools and difficult arguments, so that the additional terms may pose some problems, even if they are of lower order. Especially the term involving ∇U could cause some difficulties, because its non-local nature makes it difficult to say much about it in advance. Thus if we want to study the question of regularity for problems like this one, we need to find out how the methods can be adapted when we have variants of the harmonic maps equation.

In Chapter 4, we study solutions of the equation

$$\Delta u + A(u)(\nabla u, \nabla u) = f \quad \text{in } \Omega \tag{1.7}$$

for a function $f : \Omega \to \mathbb{R}^n$ in a certain L^p-space. In the example of the functional F, it is not difficult to show that the lower order terms of the equation are in appropriate L^p-spaces under reasonable assumptions, and the same is true for many other problems that can be encountered in applications.

It is sometimes relatively easy to generalize the methods of the theory of harmonic maps to such a situation, and sometimes very hard, depending mainly on the number p (in relation to the dimension m). We consider in particular three cases. First we study the case $p > \frac{m}{2}$ (and for technical reasons we always assume $p \geq 2$). Here we can prove Hölder continuity of solutions of (1.7) under the conditions that generalize the notion of stationary weakly harmonic maps naturally. In the border case $p = \frac{m}{2}$, continuity cannot be expected any longer; already the theory of linear equations shows that. It is still possible, however, to prove certain energy inequalities that can be useful for certain problems. Finally we study solutions of (1.7) for $p = 2$ in dimensions $m \leq 4$. If $p = 2$, the dimension four plays a similar role for the problem belonging to (1.7) as the dimension two plays for weakly harmonic maps. In particular, the problem is relatively easy—although obviously still more difficult than the corresponding problem for harmonic maps—if $m < 4$ (and hence $p > \frac{m}{2}$); it is moderately difficult if $m = 4$, and very difficult indeed if $m > 4$. We do not consider the last of these cases. In the other two cases, however, we can find some estimates for the second derivatives of solutions of (1.7).

There are many possible ways to generalize the arguments further. For instance, some of the methods we use in Chapter 4 also work for solutions of equations like

$$\Delta u + A(u)(\nabla u, \nabla u) = f + \operatorname{div} g \quad \text{in } \Omega,$$

where $g : \Omega \to \mathbb{R}^{m\times n}$ is in an appropriate L^q-space (and such equations may also appear in applications). Our goal, however, is not to give the