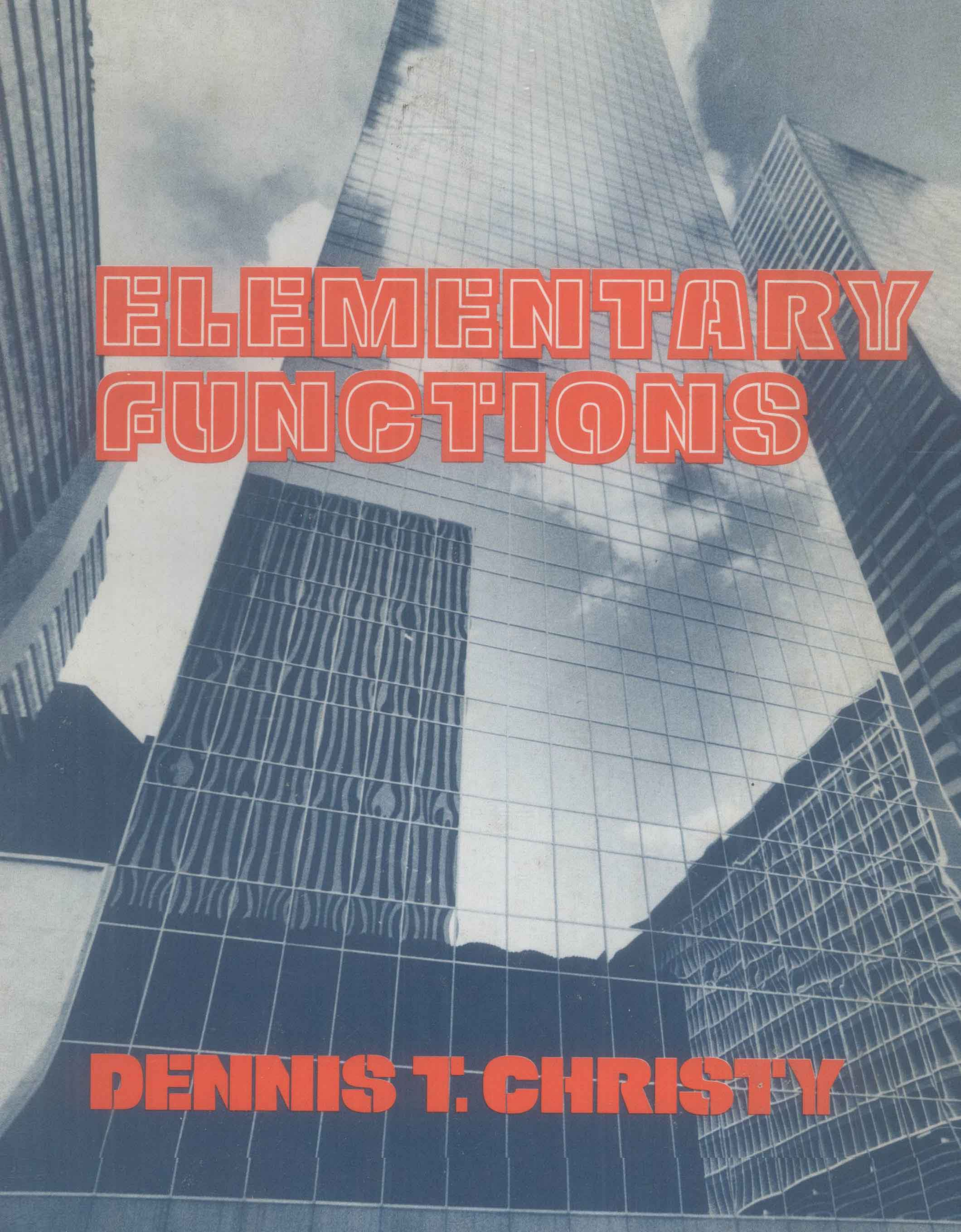

ELEMENTARY FUNCTIONS

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ELEMENTARY FUNCTIONS

PREFACE

This book is intended for students who need a concrete approach to mathematics. Set theory and notation are kept to a minimum. The text stresses problem solving and includes many applications that indicate the need for analyzing concepts with mathematical tools.

My experience has been that college students who need precalculus mathematics learn best by "doing." Examples and exercises are crucial. The text contains brief, precisely formulated paragraphs followed by many detailed examples. Each section has enough exercises both for in-class practice and for homework. The problem sets are carefully graded and contain an unusual number of routine manipulative problems. There is nothing more frustrating than being stuck on the beginning exercises! So that each student may be challenged fairly, difficult problems have been included. Above all, there is a wide variety of questions and discussions that show the student how mathematics helps us to master our world.

This text covers the usual topics of "elementary functions" and is basically the "Math 0" course recommended by CUPM. ("A Transfer Curriculum in Mathematics for Two Year Colleges," The Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America, Berkeley, California, 1969.) The function concept plays a unifying role in the study of polynomial, rational, exponential, logarithmic, circular, and trigonometric functions. Analytical geometry is integrated throughout the text and Chapter 8 discusses conic sections. Supplementary topics include sequences, series, the binomial theorem, and mathematical induction.

The presentation assumes that the student has mastered the mechanical techniques usually associated with a course in intermediate algebra. After this course the student can proceed directly to the study of calculus.

Substantial portions of this book are taken from my first publication *Essentials of Precalculus Mathematics*, and as such, this book is the culmination of work that began in 1973. Over this period many people have generously contributed their time and effort and I am indebted to all of them. Professor Frank Kocher of Pennsylvania State University deserves recognition for helping with the difficult job of making the mathematics precise without making the style and material too difficult for the intended audience. My thanks to the staff of Harper & Row, especially Cynthia Hausdorff, who skillfully guided me through the production of both of my books. I am grateful also to all my colleagues at Nassau Community College and in particular to Professors James Baldwin, Bob Rosenfeld, John Schreiber, Michael Totoro, and Gene Zirkel for their support and constructive criticism during the extensive class testing of the manuscript. Special recognition is due Professor Abraham Weinstein, our chairman, for his unwavering support. Class testing with preliminary editions of a manuscript can lead to many administrative problems. In each difficulty Professor Weinstein provided the help and leadership that were necessary to overcome the problem. To my parents a special "Thank you." You have always enthusiastically supported my efforts and this undertaking was no exception. Finally, it has been said that being married to a writer is a fate worse than death, since he is physically home and mentally absent—the worst possible combination. My thanks to my wife, typist, and proofreader, Margaret Ellen, who escapes this fate as much as possible by making my projects, her projects.

Dennis T. Christy

CONTENTS

CHAPTER 1

BASIC CONCEPTS

1–1 REAL NUMBERS

Mathematics is a basic tool for analyzing concepts in every field of human endeavor. In fact, the primary reason you have studied this subject for at least a decade is that mathematics is the most powerful instrument available to man in the search to understand our world and to control it. Mathematics is essential for full comprehension of technological and scientific advances, economic policies and business decisions, and the complexities of social and psychological issues. Calculus, statistics, and computer science are but a few of the mathematical areas which contain important applications in all these fields.

If your experience with mathematics has been limited or unfavorable, you may hestitate to devote yourself to material that seems so abstract and technical. Nevertheless, if you fail to grasp at least the rudiments of this language, you will find the development and application of important concepts difficult to comprehend. Remember, you learn mathematics by doing mathematics, or perhaps more figuratively, "Mathematics is not a spectator sport." Equally important, remember that at the heart of mathematics is algebra. A knowledge of algebraic concepts and manipulations is necessary for this course and any future work in mathematics. If your algebra is rusty the time to begin reviewing is right now!

We begin by considering the types of numbers that are needed in a technological society.

1. Our most basic need is for the numbers that are used in counting: 1, 2, 3, . . . (". . ." means "and so on"). These numbers stand for whole quantities and are the first numbers that we learn.

EXAMPLE 1: When 500 students were sampled, 417 felt that the F grade should be abolished.

2. Our practical need to make precise measurements of quantities such as length, weight, and time makes the concept of fractions and decimals familiar to all of us.

EXAMPLE 2: The width of the room is $9\frac{1}{2}$ ft. The winning time in the race is 45.9 seconds.

3. Positive and negative numbers are needed to designate direction or to indicate whether a result is above or below some reference point.

EXAMPLE 3: In physics, the velocity of an object indicates both its speed and direction. A rocket traveling at a speed of 100 ft/second has a velocity of $+100$ ft/second when it is rising and -100 ft/second when it is falling.

EXAMPLE 4: In statistics, there is a rating system that assigns a positive rating to a score above the mean (average) and a negative rating to a score below the mean.

4. In technical work, we need special numbers, such as $\sqrt{2}$, π (pi), and $\sqrt{-1}$. We will discuss these numbers later.

We now formalize our work by giving specific names to various collections of numbers. *The collection of the counting numbers, zero, and the negatives of the counting numbers is called the integers* (*see Figure 1.1*).

$$\underbrace{\ldots, -5, -4, -3, -2, -1}_{\text{negative integers}} \qquad \underbrace{0}_{\text{zero}} \qquad \underbrace{1, 2, 3, 4, 5, \ldots}_{\text{positive integers}}$$

Figure 1.1 Integers

The collection of fractions with an integer in the top of the fraction (*numerator*) *and a nonzero integer in the bottom of the fraction* (*denominator*) *is called the rational numbers.* Symbolically, a rational number is a number of the form a/b, where a and b are integers, with b not equal to ($\neq$) zero. The numbers $\sqrt{2}/3$ and $2/\pi$ are fractions. They are not rational numbers because they are not the quotient of two integers. All integers are rational numbers because we can think of each integer as having a 1 in its denominator. (Example: $4 = 4/1$.)

Our definition for rational numbers specified that the denominator cannot be zero. Let us see why.

$$\frac{8}{2} = 4 \text{ is equivalent to saying that } 8 = 4 \cdot 2$$

$$\frac{55}{11} = 5 \text{ is equivalent to saying that } 55 = 5 \cdot 11$$

If $8/0 = a$, where a is some rational number, this would mean that $8 = a \cdot 0$. But $a \cdot 0 = 0$ for any rational number. There is no rational number a such that $a \cdot 0 = 8$. Thus, we say that $8/0$ is *undefined.*

Now consider $0/0 = a$. This is equivalent to $0 = a \cdot 0$. But $a \cdot 0 = 0$ for *any* rational number. Thus, not just one number a will solve the equation—any a will. Since $0/0$ does not name a particular number, it is also undefined. Consequently, division by zero is undefined in every case. That is why the denominator in a rational number is not zero.

To define our next collection of numbers we now consider the decimal representation of numbers. We may convert rational numbers to decimals by long division. Consider the following examples of repeating decimals. A bar is placed above the portion of the decimal that repeats.

$$\frac{3}{4} = \begin{cases} 0.7500\ldots \\ \text{or} \\ 0.75\overline{0} \end{cases} \qquad \frac{2}{3} = \begin{cases} 0.6666\ldots \\ \text{or} \\ 0.\overline{6} \end{cases} \qquad \frac{8}{7} = 1.\overline{142857}$$

$$\begin{array}{r} .75\overline{0} \\ 4\overline{)\,3.00} \\ \underline{2\,8} \\ 20 \\ \underline{20} \\ 0 \end{array} \qquad \begin{array}{r} .\overline{6} \\ 3\overline{)\,2.0} \\ \underline{1\,8} \\ 2 \end{array} \qquad \begin{array}{l} 1.\overline{142857} \\ 7\overline{)\,8.000000} \\ \underline{7} \\ 1\,0 \\ \underline{\,7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

The decimals repeat because at some point we must perform the same division and start a cycle. For example, when converting $\frac{8}{7}$, the only possible remainders are

0, 1, 2, 3, 4, 5, and 6. In performing the division we had remainders of 1, 3, 2, 6, 4, and 5. In the next step we must obtain one of these remainders a second time and start a cycle, or obtain 0 as the remainder, which results in repeating zeros. Thus, if a/b is a rational number, it can be written as a repeating decimal.

It is also true that any repeating decimal may be converted to a ratio between two integers. For example, let us determine the fractional equivalent to the repeating decimal 0.1717 . . . or $0.\overline{17}$. First let $x = 0.1717\ldots$. Multiplying both sides of this equation by 100 moves the decimal two places to the right, so we obtain

$$\begin{aligned} 100x &= 17.1717\ldots \\ x &= 0.1717\ldots \\ \hline 99x &= 17 \end{aligned}$$

now subtracting yields $99x = 17$ or $x = \dfrac{17}{99}$

Thus the repeating decimal $0.\overline{17}$ is equivalent to the fraction $\frac{17}{99}$. In this example we multiplied by 100 because the decimal repeated after every two digits. If the decimal repeats after one digit, we multiply by 10; if it repeats every three digits, we multiply by 1000, and so on. In summary, we have now shown that we may define a rational number either as the quotient of two integers or as a repeating decimal.

There are decimals that do not repeat, and the collection of these numbers is called the irrational numbers.

EXAMPLE 5: The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, and $\sqrt{7}$ are irrational because they have nonrepeating decimal forms. A proof that $\sqrt{2}$ cannot be written as the quotient of two integers (and equivalently as a repeating decimal) is given in Exercise 51.

EXAMPLE 6: The number $\sqrt{4}$ is not irrational because $\sqrt{4} = 2$, which is a rational number. (*Note:* The symbol $\sqrt{\ }$ denotes the positive square root of a number. Thus, $\sqrt{4} \neq -2$.)

EXAMPLE 7: The number π, which represents the ratio between the circumference and the diameter of a circle, is a nonrepeating decimal (irrational number). The fraction $\frac{22}{7}$ is only an approximation for π ($\frac{355}{113}$ is a much better one).

Since an irrational number is a nonrepeating decimal and a rational number is a repeating decimal, there is no number that is both rational and irrational. *The collection of numbers that are either repeating decimals (rational numbers) or nonrepeating decimals (irrational numbers) constitutes the real numbers.* Real numbers are used extensively in this text as well as in calculus. All the numbers that we have mentioned (except the square roots of negative numbers, such as $\sqrt{-1}$) are real numbers. Unless it is stated otherwise, you may assume that the symbols in algebra (such as x) may be replaced by any real number. Consequently, the rules that govern real numbers determine our methods of computation in algebra. A graphical illustration of the various collections of numbers is given in Figure 1.2, where we think of a given collection of numbers as being contained in the appropriate rectangular region.

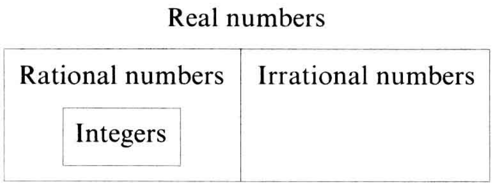

Figure 1.2

Finally, let us consider two fundamental concepts associated with real numbers: (1) how they can be represented geometrically and (2) the different ways in which real numbers replace algebraic symbols.

Real number line

The real numbers may be interpreted geometrically by considering a straight line. Every point on the line can be made to correspond to a real number and every real number can be made to correspond to a point. The first point that we designate is zero. It is the dividing point between positive and negative real numbers. Any number to the right of zero is called positive, and to the left of zero, negative. Figure 1.3 is the result of assigning a few positive and negative real numbers to points on the line.

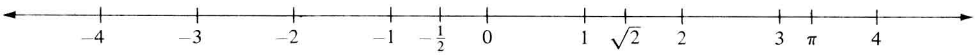

Figure 1.3

Variables and constants

In algebra, two types of symbols are used to represent numbers: variables and constants. A *variable* is a symbol that may be replaced by different numbers in a particular problem. Generally, letters near the end of the alphabet, such as x, y, or z, are variables. A *constant* is a symbol that represents the same number throughout a particular problem. Numbers, such as 2, $-\sqrt{5}$, and π, never change value and are called *absolute constants*. If we do not know the fixed number until we are given specific information about the problem, the symbol is called an *arbitrary constant*. Generally k or letters near the beginning of the alphabet, such as a, b, and c, are arbitrary constants.

EXAMPLE 8: The sales tax T on a purchase is related to the price (p) of the item by the formula $T = kp$, where k is the sales tax rate. T and p are variables that may take on different values. The sales tax rate is fixed for any particular location so that k is a constant. For example, in New York City k is fixed at 8 percent. In another location k may be fixed at a different percentage.

EXERCISES 1–1

In Exercises 1–20 classify each number by placing a check in the appropriate categories.

	Number	Real number	Rational number	Irrational number	Integer	Positive integer	None of these
Example:	14	✓	✓		✓	✓	
1.	-19						
2.	5						
3.	1						
4.	0						
5.	π						
6.	45.9						
7.	$-25/3$						
8.	$-\pi$						
9.	$\sqrt{7}$						
10.	$\sqrt{9}$						
11.	$\sqrt{-4}$						
12.	$-\sqrt{4}$						
13.	6/9						
14.	$\pi/2$						
15.	$1/\sqrt{2}$						
16.	$\sqrt{-2/3}$						
17.	$0.\overline{3}$						
18.	$9.\overline{128}$						
19.	7%						
20.	200%						

In Exercises 21–30 express each rational number as a repeating decimal.

21. $\frac{4}{5}$

22. $\frac{5}{4}$

23. $\frac{1}{3}$

24. $\frac{2}{9}$

25. $\frac{5}{11}$

26. $\frac{5}{12}$

27. $\frac{37}{6}$

28. $\frac{26}{11}$

29. $\frac{10}{7}$

30. $\frac{100}{99}$

In Exercises 31–40 express each repeating decimal as the ratio of two integers.

31. $0.\overline{2}$

32. $0.\overline{07}$

33. $0.\overline{321}$

34. $0.6\overline{332}$

35. $0.3\overline{0}$

36. $1.7\overline{0}$

37. 5.9 **38.** $4.\overline{81}$
39. 2.143 **40.** $2.1\overline{43}$

In Exercises 41–50 answer true or false. If false, give a specific counterexample.

41. All rational numbers are integers.
42. All rational numbers are real numbers.
43. All real numbers are irrational numbers.
44. All integers are irrational numbers.
45. The quotient of two integers is always an integer.
46. The quotient of two integers is always a rational number.
47. Every real number is either a rational number or an irrational number.
48. A number that can be written as a repeating decimal is called a rational number.
49. The nonnegative integers are the positive integers.
50. The nonpositive real numbers are the negative real numbers and zero.
51. The following is a proof that $\sqrt{2}$ is not a rational number. Read through it *slowly* and *carefully*. The method employed is that of *indirect proof*, one of the most powerful methods in mathematics. *We prove that $\sqrt{2}$ is not a rational number by showing that the assumption that it is a rational number leads to a contradiction.* Suppose there is a rational number equal to $\sqrt{2}$. Then it can be written as a fraction a/b, where a and b are both integers. Since the square of this fraction must equal 2, we can write $(a/b)(a/b) = 2$, or equivalently, $a^2 = 2b^2$. An important number theorem, called the *fundamental theorem of arithmetic*, states a number can be written as the product of prime factors in exactly one way if we disregard the order of the factors. For example, in terms of prime numbers we may write 12 only as $2 \cdot 2 \cdot 3$. Thus a, when expressed as the product of its primes (say n primes), can only be written as

$$a = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

Note that if a is a prime number then $n = 1$. Similarly b, when expressed as the product of its primes (say m primes), can be written only as

$$b = b_1 \cdot b_2 \cdot b_3 \cdots b_m$$

Thus we have

$$\underbrace{\underbrace{a}_{\substack{\text{Product of}\\ n \text{ primes}}} \cdot \underbrace{a}_{\substack{\text{Product of}\\ n \text{ primes}}}}_{\substack{\text{Product of } 2n \text{ or an}\\ \textit{even} \text{ number of primes}}} = \underbrace{2 \cdot \underbrace{b}_{\substack{\text{Product of}\\ m \text{ primes}}} \cdot \underbrace{b}_{\substack{\text{Product of}\\ m \text{ primes}}}}_{\substack{\text{Product of } 2m+1 \text{ or an}\\ \textit{odd} \text{ number of primes}}}$$

Our result, that an expression with an even number of prime factors equals an expression with an odd number of primes, contradicts the fundamental theorem of arithmetic. We were led to this contradiction by assuming that there is a rational number equal to $\sqrt{2}$. Therefore, we must reject this assumption. Note that any prime may replace 2 in the above proof, so we immediately have the result that *the square root of any prime number is irrational.*

As an exercise, modify this proof slightly and show $\sqrt{6}$ is irrational.

52. Show that $1 + \sqrt{2}$ is irrational by an indirect proof. (*Hint:* The result of subtracting two rational numbers is always a rational number.)

53. Explain the following paradox:

$$
\begin{array}{rl}
a = b & a, b \neq 0 \\
a^2 = b^2 & \text{square both sides of the equation} \\
a^2 - b^2 = 0 & \text{subtract } b^2 \text{ from both sides of the equation} \\
(a + b)(a - b) = 0 & \text{factor} \\
\dfrac{(a + b)\overset{1}{\cancel{(a - b)}}}{\underset{1}{\cancel{a - b}}} = \dfrac{0}{a - b} & \text{divide both sides of the equation by the same number} \\
a + b = 0 & \\
a = -b & \text{subtract } b \text{ from both sides of the equation}
\end{array}
$$

Thus, a equals both b and the negative of b!

54. Explain the following paradox, which is obtained in solving the equation $x - 2 = 1$ in the following roundabout way:

$$
\begin{array}{rl}
x - 2 = 1 & \\
x^2 - 8x + 12 = x - 6 & \text{multiply both sides of the equation by } x - 6 \\
x^2 - 8x + 15 = x - 3 & \text{add 3 to both sides of the equation} \\
\dfrac{(x - 5)\overset{1}{\cancel{(x - 3)}}}{\underset{1}{\cancel{x - 3}}} = \dfrac{\overset{1}{\cancel{x - 3}}}{\underset{1}{\cancel{x - 3}}} & \text{divide both sides of the equation by } x - 3 \\
x - 5 = 1 & \\
x = 6 & \text{add 5 to both sides of the equation}
\end{array}
$$

But 6 is not the correct solution since it does not satisfy the original equation $x - 2 = 1$ (that is, $6 - 2 \neq 1$).

1–2 PROPERTIES OF REAL NUMBERS

In order to understand the laws that govern the symbols in algebra, we must examine the properties of the numbers the symbols represent. Unless a problem states otherwise, algebraic symbols may be replaced by any real number. Thus, the rules for symbol manipulation are the same as those for the arithmetic of real numbers. Here we list some of the properties that we assume from experience for addition and multiplication of real numbers.

> PROPERTY 1: Associative property of addition: If a, b, and c are any real numbers, then $(a + b) + c = a + (b + c)$. We obtain the same result if we change the grouping of the numbers in an addition problem.

EXAMPLE 1: $(2 + 3) + 4 = 2 + (3 + 4)$

$$
\begin{aligned}
5 + 4 &= 2 + 7 \\
9 &= 9
\end{aligned}
$$