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Algebra and Its Applications

International Conference
Algebra and Its Applications
March 22–26, 2005
Ohio University, Athens, Ohio

Dinh V. Huynh
S. K. Jain
S. R. López-Permouth
Editors

with the cooperation of Pramod Kanwar

American Mathematical Society

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Algebra and Its Applications

Preface

The present volume and the conference that brought about its existence are the latest chapter in a long tradition of conferences on various facets of Algebra and its Applications hosted at the Athens campus of Ohio University. As with its predecessors in 1976, 1989, and 1999, the aim of the 2005 Algebra Conference was to bring together experts from several areas of Algebra and to provide them with an opportunity to share their research and learn from one another. We are thankful to all the outstanding mathematicians who accepted our invitation to participate in the conference and who submitted their valuable work to these Proceedings.

As a reflection of the growth and diversification of the interests of the Algebra group at Ohio University, the conferences and the proceedings volumes that followed them have continued to grow in scope. This growth has been reflected, for example, in the names of the conferences. Having started as Conferences on Ring Theory, they are now called Conferences on Algebra and its Applications. Inevitably, the topics covered in this volume reflect to some extent the interests of the editors and their colleagues.

An important development of recent years is the founding of the Center for Ring Theory and its Applications in the fall of 2000. The CRA has fueled algebraic activity in the State of Ohio and promoted cooperation between algebraists at the various campuses of Ohio University and those of Ohio State University. Most importantly, the Center has promoted such activities as this latest conference. We are much obliged to the Board of Trustees of Ohio University for following the recommendations of Ohio University presidents Robert Glidden and Roderick McDavis to provide continuing funding for CRA activities.

We wish to thank all the colleagues who served as anonymous referees; their thorough and meticulous screening of the submitted papers was invaluable for the publication of this volume. We also wish to express our appreciation to Mr. Ashish Srivastava for his help in so many ways with the running of the Conference and the publication of these Proceedings. As always, we are indebted to the staff of the American Mathematical Society for their outstanding work. Special thanks are due, in particular, to Ms. Christine Thivierge for her tireless efforts to see this project to conclusion.

The Editors

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ON COUNTABLY Σ-CS MODULES

ADEL N. ALAHMADI, HUSAIN S. AL-HAZMI, AND
PEDRO A. GUIL ASENSIO

ABSTRACT. We study a criterion for a uniform countably Σ-CS module to be (Σ-)quasi-injective. As a consequence, we get necessary and sufficient conditions that force a direct sum of indecomposable modules to be Σ-CS provided that it is countable Σ-CS.

1. INTRODUCTION

A module M over a ring R is called CS (or *extending*, see [3]) if every submodule is essential in a direct summand of M. And it is called (countably) Σ-CS module if every direct sum of (countably many) copies of M is CS. The ring R is called right Σ-CS if it Σ-CS as right R-module. Right Σ-CS rings were first studied by Oshiro under the name of co-H-rings [11]. He proved that every right Σ-CS ring is both-sided artinian and therefore, it is a direct sum of uniform right ideals. After these results, the problem of whether a Σ-CS module is also a direct sum of uniform submodules became a major problem for these modules. This question was positively answered in [6, 8, 7], where it is also shown that any module M such that $M^{(I)}$ is CS for some uncountable index set I, is actually Σ-CS. This is the best bound possible, since there exist examples of non-singular right self-injective rings R such that $R_R^{(\aleph_0)}$ is CS, but R_R is neither Σ-CS nor a direct sum of uniform ideals (see [3, Example 12.20 (i)]).

However, there are not known examples of countably Σ-CS modules or rings that are a direct sum of uniform submodules but they are not Σ-CS. This led Huynh and Rizvi to ask in [9] whether a countably Σ-CS ring (or module) that is a direct sum of uniforms might be Σ-CS. After the results in [1] (see also [4, 5]), this problem is equivalent to ask whether a uniform countably Σ-CS module must be (Σ-)quasi-injective.

In this paper, we study necessary and sufficient conditions that force a direct sum $\oplus_I M_i$ of indecomposable modules to be Σ-CS provided that it is countably Σ-CS. Our main result states that this is the case if and only if every M_i has ω_1-ACC on monomorphisms (see next section for the definition). Equivalently, if and only if the quasi-injective hull of each M_i has ω_1-ACC on submodules isomorphic to M_i.

Throughout this paper all rings R will be associative and with identity. And Mod-R will denote the category of right R-modules. By a module we will mean a unital right R-module. A submodule C of a module M is said to be *closed* in M if it has no proper essential extension in M. A submodule X of M is called a

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complement if it is maximal with respect to $X \cap Y = 0$ for some submodule Y. A module M is called *uniform* if $M \neq 0$ and any two nonzero submodules of M intersect nontrivially. Clearly, an indecomposable module is CS if and only if it is uniform. Given a module M, we will denote by $Q(M)$ its *quasi-injective hull.*

We refer to [2], [3] and [12] for any undefined notion used in the text.

2. When is a countably Σ-CS module, which is a direct sum of uniform modules, Σ-CS ?

We begin by stating a couple of results which will be used in this section.

Proposition 1. ([1], Proposition 3.5) Let $\{M_i\}_{i\in I}$ be a family of indecomposable modules such that $M = \oplus_{i\in I} M_i$ is countably Σ-CS. The following conditions are equivalent:

(i) M is Σ-CS;
(ii) Each M_i is Σ-CS;
(iii) Each M_i is (Σ-)quasi-injective;
(iv) Each M_i has local endomorphism ring;
(v) Each M_i has ACC (DCC) on submodules isomorphic to M_i.

The following key lemma was used in [7] to show that any Σ-CS module is a direct sum of uniform submodules. We denote by $N \subseteq_e M$ the fact that N is an essential submodule of M.

Lemma 2. *([7], Lemma 2.1) Let M be a CS module and $f : M \to N$ be an epimorphism. If there exists a submodule $L \subseteq M$ such that $L \cap Ker(f) = 0$ and $f(L) \subseteq_e N$, then Ker(f) is a direct summand of M.*

Given a right R-module M, let us denote by *Add*$[M]$ the full subcategory of Mod-R whose objects are the direct summands of arbitrary direct sums of copies of M. And we are going to say that a module N_R is $\aleph$-M-generated, for some cardinal number $\aleph$, if there exists an epimorphism $M^{(\aleph)} \to N$. Our first Lemma is an easy consequence of Lemma 2.

Lemma 3. *Let M be a countably Σ-CS uniform right R-module. Then every countably M-generated submodule of $Q(M)$ containing M belongs to Add$[M]$.*

Proof. Let L be a countably M-generated submodule of $Q(M)$ containing M. Then there exists an epimorphism $f : \oplus_{\alpha\in A} M_\alpha \to L$ with $|A| = \aleph_0$ and $M_\alpha = M$ for any $\alpha \in A$. Replacing if necessary f by the homomorphism $g : M \oplus (\oplus_A M_\alpha) \to L$ induced by the inclusion $u : M \to L$ and f, we can assume that there exists an $\alpha_0 \in A$ such that $f\mid_{M_{\alpha_0}} = u$. Thus, $f\mid_{M_{\alpha_0}}$ is one-to-one and $f(M_{\alpha_0}) = M$ is an essential submodule of L. Applying Lemma 2, we deduce that $Ker(f)$ is a direct summand of $\oplus_{\alpha\in A} M_\alpha$. Hence, L is isomorphic to a direct summand of $\oplus_{\alpha\in A} M_\alpha$. □

Corollary 4. *Let P be a projective countable Σ-CS module. Then the quasi-injective hull of P is flat. In particular, the injective envelope of any right countably Σ-CS ring is flat.*

Proof. The quasi-injective envelope of P is the directed union of its finitely generated submodules. Say $Q(P) = \cup_I N_i$. Let us fix an N_i. As $Q(P)$ is P-generated and N_i is finitely generated, there exists an epimorphism $p : L \to N_i + P$, where L

is a submodule of $P^{(n)}$, for some n. This epimorphism extends to a homomorphism $f : P^n \to Q(P)$, since $Q(P)$ is quasi-injective. Set $X = Im(f)$. Then $X \in Add[P]$ is projective by the above Lemma. This means that $Q(P)$ is the directed union of projective submodules and therefore, flat. □

Corollary 5. *Let $M = \oplus_{i\epsilon I} M_i$ be a countably Σ-CS module with each M_i indecomposable. Then M is Σ-CS if and only if $Q(M_i)$ is $\aleph_0$-M_i-generated for each $i \in I$.*

Proof. Let $i \in I$ and call $N = M_i$. Suppose that $Q(N)$ is $\aleph_0$-N-generated. By Lemma 3, $Q(N) \in Add[N]$. So there exists a splitting epimorphism $p : N^{(A)} \to Q(N)$ for some index set A. Let $w : Q(N) \to N^{(A)}$ such that $p \circ w = 1_{Q(N)}$. Let $x \in N$ be a non-zero element. Then $w(x)$ embeds in $N^{(B)}$ for some finite subset $B \subseteq A$. Moreover, xR is uniform since it is a submodule of N that is indecomposable and CS. Therefore there must exist a projection $\pi_a : N^{(A)} \to N_a$ over some coordinate $N_a = N$ of $N^{(A)}$ such that $\pi_a \circ w \mid_{xR}$ is a monomorphism. Hence $\pi_a \circ w$ must also be a monomorphism, because xR is essential in $Q(N)$. And, as $Q(N)$ is N-injective, this means that $Q(N)$ is isomorphic to a direct summand of $N_a = N$. Therefore, $N = Q(N)$. Finally, M is Σ-CS by Proposition 1. The converse also follows from Proposition 1. □

Next, we define the notion of γ-ACC on monomorphisms for a module M, where γ is an ordinal number.

Definition 6. *Let M be a module and γ, an ordinal number. We will say that M has γ-ACC on monomorphisms if every directed system of proper monomorphisms $\{f_{\alpha\beta} : M_\alpha \to M_\beta\}_{\alpha\leq\beta<\delta}$ with $M_\alpha \cong M$ for every $\alpha < \delta$, satisfies that $\delta < \gamma$.*

Obviously, every module M satisfies γ-ACC on monomorphisms for any ordinal γ with $| \gamma | > | M |$. We are now ready to state the main result of this paper. Let us denote by ω_0 the first infinite ordinal number and ω_1 the first uncountable ordinal number.

Theorem 7. *Let $M = \oplus_{i\in I} M_i$ be a direct sum of uniform modules and suppose that M is countably Σ-CS. The following conditions are equivalent:*

(1) *M is Σ-CS.*
(2) *M_i has ω_1-ACC on monomorphisms for each $i \in I$.*
(3) *M_i satisfies ω_1-ACC on submodules isomorphic to M_i for each $i \in I$.*

Proof. 1) ⇒ 2). This is clear since in this case each M_i is quasi-injective by Proposition 1 and therefore, any monomorphism $M_i \to M_i$ is splitted.

2) ⇒ 3) This is obvious.

3) ⇒ 1). By Proposition 1, it is enough to prove that each M_i has local endomorphism ring. Assume on the contrary that $End(M_i)$ is not local, for some $i \in I$, and call $N = M_i$. Then there exists an endomorphism $f : N \to N$ such that neither f nor $1 - f$ is an isomorphism. As $Ker(f) \cap Ker(1 - f) = 0$ and N is uniform, f or $1 - f$ must be a monomorphism. Replacing f by $1 - f$ if necessary, we may assume that f is a monomorphism.

Let us fix, for any $\alpha < \omega_1$, $M_\alpha = N$ and $f_{\alpha\alpha} = 1_N$. We are going to construct a family of submodules $\{L_\alpha\}_{\alpha<\omega_1}$ of $Q(M)$ isomorphic to N satisfying that:

- $L_\alpha \subsetneq L_\beta$ for any $\alpha < \beta < \omega_1$
- The family $\{M_\alpha\}_{\alpha\leq\gamma}$ is directed, for any $\gamma < \omega_1$.

Let us note that if we make our construction, we are done since this contradicts that $Q(N)$ has ω_1-ACC on submodules isomorphic to N.

Let us construct the family $\{L_\alpha\}_{\alpha<\omega_1}$ by transfinite induction on $\gamma < \omega_1$.

(1) First, assume that $\gamma = \delta + 1 < \omega_1$ is a successor ordinal and that we have constructed L_α for $\alpha < \gamma$ satisfying the above hypotheses. As $Q(N)$ is quasi-injective, the monomorphism $f : L_\delta = N \to N$ extends to a homomorphism $g : N \to Q(N)$ such that $g \circ f$ is the inclusion of N in $Q(N)$. And g is a monomorphism since N is uniform. Let us call $L_\gamma = Im(g)$. Then $L_\gamma \cong N$. Moreover $L_\delta \subsetneq L_\gamma$ because otherwise f would be an isomorphism.

(2) Next, assume that $0 < \gamma < \omega_1$ is a limit ordinal and we have defined L_α for any $\alpha < \gamma$. We know that for every ordinal $\delta < \gamma$, the family $\{L_\alpha\}_{\alpha\leq\delta}$ is directed. Therefore, the family $\{L_\alpha\}_{\alpha<\gamma}$ is also directed. Let $L = \varinjlim L_\alpha \subseteq Q(N)$ be the directed union of this system. As $\gamma < \omega_1$, it is countable, so we know by Lemma 3 that $L \in Add[N]$. Using now the same reasoning as in Corollary 2, we deduce that there exists a monomorphism $w : L \to N$. This homomorphism w extends to a monomorphism $h : N \to Q(N)$, because $Q(N)$ is quasi-injective and L is uniform. Let us then call $L_\gamma = Im(h)$. Clearly $\{L_\alpha\}_{\alpha\leq\gamma}$ is directed since $\{L_\alpha\}_{\alpha<\gamma}$ is directed, $L = \varinjlim L_\alpha$ and $L \subseteq L_\gamma$. Finally, $L_\alpha \neq L_\gamma$ for any $\alpha < \gamma$ because otherwise we would have that $L_\alpha \subsetneq L_{\alpha+1} \subseteq L_\gamma = L_\alpha$.

□

Our next result shows that the endomorphism ring of a finitely generated uniform countably Σ-CS module satisfies a restricted DCC condition on cyclic left ideals. We do not know whether these endomorphism rings satisfy DCC on cyclic left ideals in general. Of course, if this would be true, these endomorphism rings would be right perfect and therefore, every finitely generated countably Σ-CS module that is a direct sum of uniforms would be Σ-CS by Proposition 1.

Proposition 8. *Let M be a uniform countably Σ-CS finitely generated module and let $S = End(M)$. Then every descending chain of cyclic left ideals $\{S\phi_n\}_{n<\omega_0}$ of S, with ϕ_n a monomorphism for every n, stabilizes.*

Proof. Assume on the contrary that $\{S\phi_n\}_{n<\omega_0}$ is a strict descending chain of left ideals with ϕ_n, a monomorphism for each n. Then there must exist monomorphisms $f_n : M \to M$ such that $\phi_n = f_n \circ \ldots \circ f_1$ for every n.

For each $i < \omega_0$, set $M_i = M$ and $N = \oplus M_i$. Let us call $\varepsilon_i : M_i \to N$ the structural injection of M_i in the direct sum. Let us define $g_i : M_i \to N$ by $g_i = \varepsilon_i - \varepsilon_{i+1} f_i$ and $g : N \to N$ by $g = \oplus g_i$. For every $i, j < \omega_0$ with $i < j$, let us set $\sigma_{ii} = 1$ and $\sigma_{ij} = f_{j-1} f_{j-2} \ldots f_{i+1} f_i$. Then $\{\sigma_{ij} : M_i \to M_j\}_{i\leq j}$ is a directed system of essential monomorphisms. Moreover, $\varinjlim M_i = N/Im(g)$ (see e.g. [12, 24.2])). Let $\pi : \oplus_{\omega_0} M_i \to M/Im(g)$ be the canonical projection. As $\{\sigma_{ij} : M_i \to M_j\}_{i\leq j}$ is a directed system of essential monomorphisms, we conclude that $\pi \circ \varepsilon_i : M_i \to \varinjlim M_i$ is an essential monomorphism for every $i \in \omega_0$. In particular, $\pi \circ \varepsilon_1 : M_1 \to \varinjlim M_i$ is an essential monomorphism. Hence, $\pi(\varepsilon_1(M_1)) \subseteq_e \varinjlim M_i$ and π restricted to $\varepsilon_1(M_1)$ is a monomorphism. Therefore, $Ker(\pi) \cap \varepsilon_1(M_1) = 0$ and $Ker(\pi)$ is a direct summand of N by Lemma 2. But $Ker(\pi) = Im(g)$ by [12, 43.3]. Thus, $Im(g)$ is a direct summand of N. Now, since M is finitely generated, there exists an n and an $h \in S$ such that $f_n f_{n-1} \cdots f_1 = h f_{n+1} f_n \cdots f_1$ by [12, 43.3(3)].

Hence, $\phi_n = h\phi_{n+1}$. And this implies that $S\phi_n = S\phi_{n+1}$. A contradiction that proves the proposition. □

As we said before, we do not know whether this DCC condition implies that any finitely generated uniform countably Σ-CS module has local endomorphism ring. We finish the paper by proving that this is the case if we assume it on right ideals instead of left ideals.

Proposition 9. *Let M be a uniform right R-module and $S = End_R(M)$. If S has DCC on cyclic right ideals generated by monomorphisms, then S is local.*

Proof. Assume to the contrary that S is not local. Then there exists an $f \in S$ such that neither f nor $1 - f$ is an isomorphism. Since M is uniform and $Ker(f) \cap Ker(1 - f) = 0$, we have that either f or $1 - f$ is monomorphism. Say that f is a monomorphism. We have the following descending chain of cyclic right ideals generated by monomorphism: $fS \supseteq f^2S \supseteq ... \supseteq f^nS....$ By assumption, there exists an $n \in \omega_0$ such that $f^nS = f^{n+1}S$. This means there exists an $h \in S$ such that $f^n = f^{n+1}h$. Let us fix an arbitrary $m \in M$. Then $f^n(m) = [f^{n+1}h](m)$. This means $f^n(m) = f^n(fh(m))$. Since f^n is monomorphism we conclude that $m = fh(m)$. Hence $m \in Im(f)$. But m was an arbitrary element of M. Therefore $f \circ h = 1_M$. And, as M is uniform, this means that f must be an isomorphism, a contradiction that finishes our proof. □

Corollary 10. *Let M be an indecomposable countably Σ-CS module and $S = End(M)$. If S has DCC on cyclic right ideals generated by monomorphisms, then M is Σ-CS.*

Proof. This is a straightforward consequence of Proposition 1 and Proposition 9. □

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Moduli spaces of graded representations of finite dimensional algebras

E. Babson, B. Huisgen-Zimmermann, and R. Thomas

ABSTRACT. Let Λ be a basic finite dimensional algebra over an algebraically closed field, presented as a path algebra modulo relations; further, assume that Λ is graded by lengths of paths. The paper addresses the classifiability, via moduli spaces, of classes of graded Λ-modules with fixed dimension d and fixed top T. It is shown that such moduli spaces exist far more frequently than they do for ungraded modules. In the local case (i.e., when T is simple), the graded d-dimensional Λ-modules with top T always possess a fine moduli space which classifies these modules up to graded-isomorphism; moreover, this moduli space is a projective variety with a distinguished affine cover that can be constructed from quiver and relations of Λ. When T is not simple, existence of a coarse moduli space for the graded d-dimensional Λ-modules with top T forces these modules to be direct sums of local modules; under the latter condition, a finite collection of isomorphism invariants of the modules in question yields a partition into subclasses, each of which has a fine moduli space (again projective) parametrizing the corresponding graded-isomorphism classes.

1. Introduction

Let Λ be a finite dimensional algebra with radical J over an algebraically closed field K, and fix a finite dimensional semisimple (left) Λ-module T together with a positive integer d. In [**4**], the second author explored the existence and structure of moduli spaces classifying, up to isomorphism, those d-dimensional (left) representations M of Λ whose tops M/JM equal T under identification of isomorphic semisimple modules. The vehicle for tackling this classification problem is a projective variety, $\mathfrak{Grass}_d^T$, parametrizing the d-dimensional modules with top T; see Section 2.

The general goal driving such investigations is to demonstrate that, even over a wild algebra Λ, major portions of the representation theory may behave tamely,

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being accessible to classification in a quite stringent sense. The idea of *moduli* goes back to Riemann's 1857 classification of nonsingular projective curves of fixed genus in terms of continuous structure determining invariants; it was made precise by Mumford in the 1960's. In rough terms, adapted to our representation-theoretic context, it amounts to the following: First one introduces a concept of *family*, which pins down what it means that a collection of finite dimensional representations be parametrized "continuously" by the points of a variety X. Given this prerequisite, a fine or coarse moduli space for a class of finite dimensional representations of Λ is a variety that continuously and bijectively parametrizes the isomorphism classes of the considered representations in a fashion satisfying a certain *coarse* or *fine* universal property. In slightly more precise language, a *fine moduli space* – the crucial concept in this paper – is the parametrizing variety of a distinguished family satisfying the postulate that any other family of representations recruited from the given class be uniquely "induced" by the distinguished one. Existence provided, both fine and coarse moduli spaces are unique up to canonical isomorphism due to the pertinent universality conditions.

Here we focus on a *graded* basic finite dimensional algebra Λ and address two problems closely related to the one mentioned at the outset: (1) That of deciding classifiability of the *graded* d-dimensional left Λ-modules M with fixed top T, up to graded-isomorphism; and, more restrictively, classifiability of those graded candidates M which have fixed *radical layering* $\mathbb{S}(M) = \left(J^l M / J^{l+1} M\right)_{l \geq 0}$. In either case, "classifiability" stands for existence of a fine or coarse moduli space. (2) In case existence of a moduli space is secured, the problem of determining the structure of this space and of constructing a universal family for the considered class of representations.

Our base field K being algebraically closed, we may assume without loss of generality that Λ is a path algebra modulo relations, meaning that $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra KQ. In fact, we specialize to the situation where Λ is *graded by lengths of paths*, meaning that I is a homogeneous ideal with respect to the natural grading of KQ through path lengths. We will start by showing that the set of those points in the mentioned variety $\mathfrak{Grass}_d^T$, which correspond to the graded d-dimensional modules with top T that are generated in degree zero, form a closed – and hence projective – subvariety of $\mathfrak{Grass}_d^T$, denoted by Gr-$\mathfrak{Grass}_d^T$. More strongly, we will verify the following: Suppose $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ is a sequence of semisimple modules with $\mathbb{S}_0 = T$, where $J^{L+1} = 0$ and the dimensions of the $\mathbb{S}_l$ add up to d. Then the following subset Gr-$\mathfrak{Grass}(\mathbb{S})$ of $\mathfrak{Grass}_d^T$ is closed: Namely, the set of those points in Gr-$\mathfrak{Grass}_d^T$ which correspond to the graded modules M with $\mathbb{S}(M) = \mathbb{S}$. In alternate terms, Gr-$\mathfrak{Grass}(\mathbb{S})$ is a projective variety parametrizing the d-dimensional graded modules generated in degree zero with radical layering $\mathbb{S}$. Closedness of these subvarieties entails, in particular, that each Gr-$\mathfrak{Grass}(\mathbb{S})$ is a union of irreducible components of Gr-$\mathfrak{Grass}_d^T$. This is the first crucial difference between the graded and ungraded settings. Indeed, by contrast, the subvariety $\mathfrak{Grass}(\mathbb{S})$ consisting of *all* points in $\mathfrak{Grass}_d^T$ corresponding to (not necessarily graded) modules with radical layering $\mathbb{S}$ fails to be closed in $\mathfrak{Grass}_d^T$ in general.

Naturally, the *graded* d-dimensional representations with top T possess a fine/ coarse moduli space whenever all d-dimensional representations with top T do. On the other hand, not too surprisingly, existence of a moduli space is a far more frequent event in the graded than in the ungraded situation, as a grading accounts for increased rigidity. What is surprising is the extent of this discrepancy: For instance, given a simple module T with projective cover P, the d-dimensional top-T modules have a fine (equivalently, a coarse) moduli space classifying them up to isomorphism precisely when every submodule C of JP having codimension d in P is invariant under endomorphisms of P; the latter requirement imposes strong restrictions on the underlying triple (Λ, T, d); see [**4**, Corollary 4.5]. However, when one narrows one's view to graded representations under graded-isomorphism, existence of a fine moduli space is automatic for a simple top T:

THEOREM A. *If Λ is path-length-graded and T a simple left Λ-module, then, for any positive integer d, the graded d-dimensional Λ-modules with top T possess a fine moduli space, classifying their graded-isomorphism classes. This moduli space equals* Gr-$\mathfrak{Grass}_d^T$.

Calling a module *local* if it has a simple top, we will more generally prove the following:

THEOREM B. *Suppose that Λ is path-length-graded, $T \in \Lambda$-mod any semisimple Λ-module, and $\mathbb{S}$ a sequence of semisimple Λ-modules as above. Moreover, let $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) be the class of all graded d-dimensional Λ-modules with top T (resp. with radical layering $\mathbb{S}$). Then the following are true:*

- *If there is a coarse moduli space classifying the graded-isomorphism classes in $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$), then every object in $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) is a direct sum of local modules.*
- *Conversely, if $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) consists of direct sums of local modules, then $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) can be partitioned into finitely many subclasses, each of which has a fine moduli space.*

All moduli spaces arising in the latter case are projective.

In parallel with the ungraded situation, each of the varieties Gr-$\mathfrak{Grass}_d^T$ possesses a distinguished affine cover, accessible from quiver and relations of Λ, which provides the key to analyses of concrete examples.

This leaves the question of which projective varieties occur among the irreducible components of fine moduli spaces for graded modules with fixed dimension and top. We use examples of Hille in [**2**], which are in turn based on a construction technique introduced by the second author in [**3**], to show that every irreducible projective variety arises as an irreducible component of such a space.

Our approach to moduli problems for representations is fundamentally different from that of King in [**7**], where the targeted modules are those that are semistable with respect to a given additive function $\Theta : \mathrm{K}_0(\Lambda\text{-mod}) \to \mathbb{R}$. King's definition of semistability allows for the adaptation of techniques developed by Mumford with the aim of classifying vector bundles. On one hand, in King's approach (coarse) moduli spaces for Θ-semistable representations are guaranteed to exist. On the other hand, in general these classes of modules are hard to assess in size and to describe in more manageable terms, while their classification through moduli spaces is a priori only up to an equivalence relation considerably coarser than isomorphism.

Concerning strategy: Evidently, every local graded module is generated in a single degree, which, for purposes of classification, we may assume to be zero. As for the general case, we will show that classifiability up to graded-isomorphism (through a moduli space) of the graded d-dimensional modules with fixed top T, generated in mixed degrees, forces these graded objects to be direct sums of local graded submodules. We are thus led back to a situation in which restriction to graded modules generated in degree zero is harmless. The proof of this reduction step requires an extra layer of technicalities likely to obscure the underlying ideas; we will therefore defer it to an appendix (Section 6). In Sections 2–5, we will only consider graded modules generated in degree zero.

In Section 2, we will provide prerequisites; in particular, we will introduce the varieties Gr-$\mathfrak{Grass}_d^T$ and Gr-$\mathfrak{Grass}(\mathbb{S})$ and verify their projectivity. In Section 3, we will prepare for proofs of the main results by introducing the mentioned affine cover of the variety Gr-$\mathfrak{Grass}_d^T$ and by constructing a pivotal family of graded modules with top T; this family will turn out to be the universal one (see Section 2 for a definition) in case a fine moduli space exists. Section 4 contains proofs of upgraded versions of Theorems A and B, the latter restricted to graded modules generated in degree zero. Section 5 is devoted to examples. The appendix, finally, will remove the restriction concerning degree-zero generating sets from the results for nonlocal modules.

2. Further terminology and Background

We will be fairly complete in setting up our conventions, even fairly standard ones, for the convenience of the reader whose expertise lies at the periphery of the subject.

Let Λ be a basic finite dimensional algebra over an algebraically closed field K. Without loss of generality, we assume Λ to be a path algebra modulo relations, that is, $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra KQ.

Gradings. Throughout, we suppose Λ to be graded in terms of path lengths, meaning that I is homogeneous with respect to the length-grading of KQ. Denoting by J the Jacobson radical of Λ, we let L be maximal with $J^L \neq 0$. Then the grading of Λ takes on the form $\Lambda = \bigoplus_{0 \le l \le L} \Lambda_l$, where $\Lambda_l \cong J^l/J^{l+1}$ is the homogeneous component of degree l of Λ. The vertices $e_1, \dots, e_n$ of Q will be identified with the primitive idempotents of Λ corresponding to the paths of length zero, that is, the e_i will also stand for the I-residues of the paths of length 0 in Λ_0. The factor modules $S_i = \Lambda e_i / J e_i$ then form an irredundant set of representatives for the simple left Λ-modules; unless we explicitly state otherwise, we consider the S_i – and hence all semisimple modules – as homogeneous modules in degree 0, systematically identifying isomorphic semisimple modules. Clearly, the grading of any indecomposable projective module Λe_i inherited from that of Λ yields a graded local module which is generated in degree zero. Whenever $P = \bigoplus_{1 \le i \le n} (\Lambda e_i)^{t_i}$, we let $P = \bigoplus_{0 \le l \le L} P_l$ be the resulting decomposition into homogeneous subspaces. Given two graded modules M, M', we call a morphism $f : M \to M'$ *homogeneous of degree* s in case $f(M_l) \subseteq M'_{l+s}$ for all l; the attribute "homogeneous" by itself stands for "homogeneous of degree zero". Whenever there is an isomorphism $M \to M'$ which is homogeneous of some degree s, we call M and M' *graded-isomorphic*; so, in particular, two graded modules generated in degree zero are graded-isomorphic if and only if they are isomorphic by way of a homogeneous map.

Paths in Λ *and top elements of modules.* We will observe the following conventions: The product pq of two paths p and q in KQ stands for "first q, then p"; in