College Algebra with Applications

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Preface

The following four points present the major intentions and features of this book:

1. This book is intended for students taking a terminal course in algebra as well as for those who plan to pursue careers in engineering, technology, mathematics, and the natural and social sciences. This text is designed to fulfill two important purposes:
   a. To present algebra as a logical system.
   b. To provide the fundamental algebraic skills and applications necessary for the student to fulfill his individual needs. This means that considerable flexibility in the choice of materials will be provided.
2. The following features may help in the choice of this book as a text in college algebra.
   a. Each chapter contains an overall statement of purpose.
   b. Each section contains a list of behavioral objectives as they apply to the material in that section.
   c. The format has been designed to emphasize the important parts and ideas being presented.
   d. The exposition is written in clear and simple language which we believe can be understood easily by the student.
   e. Each topic is amply illustrated by many solved problems which have been selected carefully to illustrate the principles being presented. These solved problems contain step-by-step, easy-to-follow solutions with the use of visual aids as needed to help in understanding the problem. In fact, many students should find that the materials presented in this book are largely self-teaching.
   f. We believe that the presentation in Chapter 4 of a structural approach to the analysis of "word problems" is the outstanding contribution of this text. The main purpose of the analysis structure is to bridge the "gap" between the words of the problem and the equation to be solved.
   g. We also believe that the extent and the variety of applications, the extent of section exercises, and the extent of chapter review problems are important contributions of this book.

h. Another important feature is the inclusion of full-page mathematical art drawings at the beginning of each chapter. Conceived by S. Al-hadad, each of these illustrations represents some of the mathematical concepts discussed in the chapter. Brief explanatory notes on the chapter opening illustrations are on page 482.

3. Additional features of the text include the following.
   a. The notion of set is introduced in Chapter 1. This is done to provide some concise language to be used throughout the book for better clarification of certain concepts. This is followed by a discussion of the axioms of equality, the field axioms, and the axioms of order as they apply to real numbers. Certain theorems based on these axioms are stated and proved. They provide the justification for the fundamental operations on algebraic expressions used later in the text.
   b. Chapters 4 and 5 discuss functions and their graphs. In Chapter 4 we present the structure for "word problem" analysis, which is used extensively and completely throughout the remainder of the book.
   c. A slightly different approach to matrices and determinants is presented in Chapter 6.
   d. Our discussion of complex numbers in Chapter 9 is unique in its presentation. Its use of the polar representation of the complex number along with some *simple* electrical applications makes this presentation of complex numbers out of the ordinary for college algebras.
   e. Chapter 10 provides a more complete presentation of polynomials (the theory of equations) than is usually given.
   f. Included in Chapter 11 is a somewhat unique presentation of mathematical induction.
4. We believe that this is a versatile text which can be used for various courses; for example:
   a. A three-unit one-quarter or one-semester course for the student with sufficient background.
   b. A five-unit one-quarter or one-semester course or a full-year course for the student needing a somewhat slower approach.

SABAH AL-HADAD
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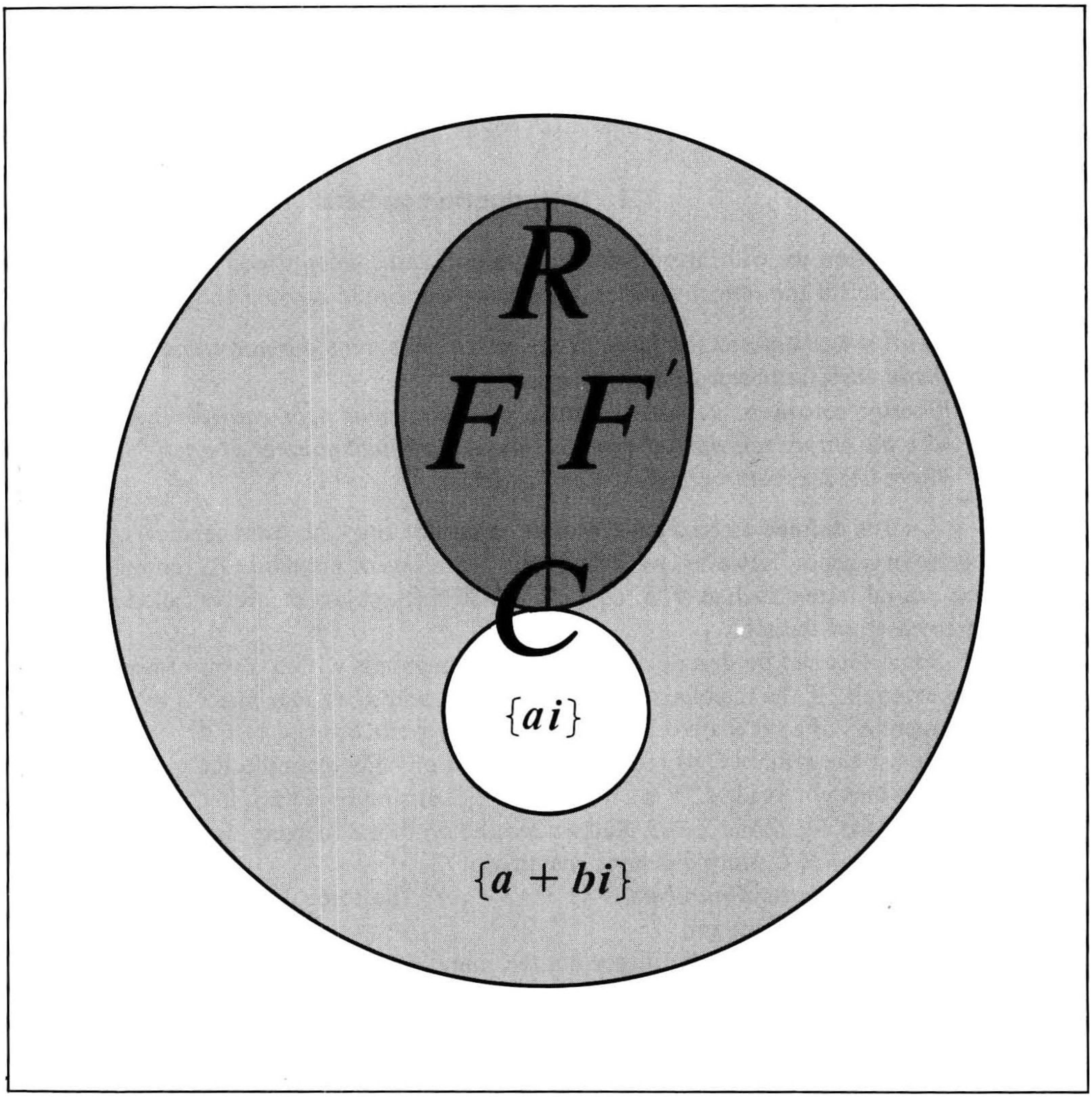
Contents

Chapter 1

Sets and Real Numbers



Without a thorough understanding of the fundamentals, no mathematical subject can be mastered or applied effectively. Accordingly, we will begin by discussing the basic symbols, terms, and assumptions of algebra.

1.1 Introduction to Sets

In this section we will introduce some symbols and concepts used when working with sets. To fulfill the objectives of this section you should be able to do the following:

a. Identify well-defined sets given in the roster and set-builder notations.
b. Distinguish finite sets from infinite sets.
c. Identify constants, variables, subsets of a set, equal sets, equivalent sets, disjoint sets, the empty set, the universal set, and the cardinal number of a set.
d. Represent set relationships by Venn diagrams.

A *set* is defined to be a collection of objects of any kind: numbers, cars, names, or tigers. In algebra, however, we deal chiefly with sets of numbers. To represent sets we use capital letters such as A, B, C, X, Y, and Z. Each object in a set is called a *member*, or *element*, of the set.

Some sets can be described by listing the members within a pair of braces, $\{\cdots\}$. For example, if the members of a set A are 1, 2, and 3, we may write $A = \{1, 2, 3\}$. If the members of a set B are a, b, c, and d, we may write $B = \{a, b, c, d\}$.

We use the symbol $\in$ to mean "is a member of." For example, if $C = \{1, 2, 5\}$, then $5 \in C$, which you read as "5 is a member of C," or simply as "5 is in C." Of course it is also true that $1 \in C$ and $2 \in C$. And we use the symbol $\notin$ to mean "is not a member of." For instance, $7 \notin C$, since 7 is not a member of C.

The set of *natural numbers* is $\{1, 2, 3, \ldots\}$. The three dots indicate that the pattern continues without end.

A *finite set* is a set that has a limited number of members. For example, the set $\{1, 2, 3, 7, 0, 23\}$ is a finite set.

An *infinite set* is a set that is not finite. For instance, the set of natural numbers is a infinite set, and we say that it has infinitely many members.

So far we have described sets by listing their members within braces. That is called the *roster notation*. Another useful method of describing sets is the *set-builder notation*. For example, the set of natural numbers can be indicated by

$$\{x|x \text{ is a natural number}\},$$

which is read as

"the set of all x such that x is a natural number."

In the set-builder notation the symbol | stands for "such that" and the letter x is a *variable*, which represents an unspecified member of the set. Of course any other letter can be used as a variable. A *constant* is a symbol that represents just one object. For example, the symbol 3 is a constant.

Example 1 Express the set $\{1, 3, 5\}$ in set-builder notation.

Solution Since 1, 3, and 5 are the only odd natural numbers that are less than 6, we can describe the set as follows:

$$\{x|x \text{ is an odd natural number less than } 6\}.$$

Example 2 Express the set $\{x|x + 3 = 5\}$ in roster notation.

Solution Since 2 is the only value of x for which $x + 3 = 5$, the set has only one number, 2. So the roster notation for the set is $\{2\}$.

A set must be *well-defined*. This means that there must be no doubt about which objects are members and which are not.

Example 3 Is the following set well-defined?

$$\{x|x \text{ is an outstanding lawyer}\}$$

Answer No, because there is no definite way to decide which lawyers are outstanding and which are not.

Example 4 Is the following a well-defined set?

$$\{x|x \text{ is an even natural number less than } 9\}$$

Answer Yes, because we know definitely that the only members are 2, 4, 6, and 8.

Many mathematicians find it convenient to use "iff" as an abbreviation of "if and only if." Thus, a statement such as "P iff Q" means "P if and only if Q," which in turn means "if P then Q, and if Q then P."

Sets A and B are *equal*, written $A = B$, iff they have the same members, that is, iff every member of A is a member of B AND every member of B is a member of A.

Example 5 Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 1\}$. Are A and B equal sets?

Answer Yes, because they have the same members: 1, 2, and 3. So we may write $A = B$.

Example 6 Let $C = \{2, 3, 4, 5\}$ and $D = \{x \mid x$ is a natural number between 1 and 6$\}$. Is it true that $C = D$?

Answer Yes, because they have the same members: 2, 3, 4, and 5.

Example 7 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Is it true that $A = B$?

Answer No, because 4 is a member of B but not of A. So we write $A \neq B$, where the symbol $\neq$ means "is not equal to."

A set A is a *subset* of a set B, written $A \subseteq B$, iff every member of A is a member of B. Note that the symbol $\subseteq$ means "is a subset of."

Example 8 Let $A = \{3, 4, 5\}$ and $B = \{2, 3, 4, 5\}$.
a. Is A a subset of B? b. Is B a subset of A?

Answers
a. Yes, because every member of A is a member of B. Hence we may write $A \subseteq B$.
b. No, because 2 is a member of B but not of A. We write $B \nsubseteq A$, where the symbol $\nsubseteq$ means "is not a subset of."

Example 9 Let $A = \{3, 6, 11\}$. Is it true that $A \subseteq A$?

Answer Yes, since every member of A is surely a member of A.

The preceding example illustrates the fact that every set is a subset of itself.

A set A is a *proper subset* of a set B, written $A \subset B$, iff A is a subset of B and B has at least one member that is not in A. Note that the symbol $\subset$ means "is a proper subset of." Of course, the symbol $\not\subset$ means "is not a proper subset of."

Example 10 Let $A = \{3, 4, 5\}$ and $B = \{2, 3, 4, 5, 7\}$.
a. Is the statement $A \subset B$ true or false?
b. Is the statement $B \not\subset B$ true or false?

Answers
a. True. A is a proper subset of B because A is a subset of B and B has at least one member that is not in A.
b. True. B is not a proper subset of B. Indeed, no set is a proper subset of itself.

Example 11 Let $A = \{3, 4\}$.
a. Is the statement $3 \subset A$ true or false?
b. Is the statement $3 \in A$ true or false?

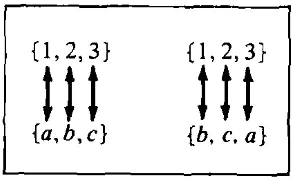
Answers
a. False! 3 is not a proper subset of A, since 3 is not a set. The symbol $\subset$ must only be used between sets.
b. True. 3 is a member of A. Remember that the symbol $\in$ means "is a member of."

We say that a *one-to-one correspondence* exists between two sets A and B if it is possible to associate the members of A and B in such a way that each member of A is associated with exactly one member of B, and each member of B is associated with exactly one member of A.

Example 12 Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Does there exist a one-to-one correspondence between the sets A and B?

Answer Yes. Figure 1 illustrates two one-to-one correspondences between A and B. (There are four other one-to-one correspondences, which you should find for yourself.)

Figure 1



Two sets are *equivalent* iff there exists a one-to-one correspondence between the sets. For instance, the sets A and B in Example 12 are equivalent.

The *cardinal number* of a finite set A is the number of members in the set. The cardinal number of a set A is indicated by the symbol $n(A)$. For example, if $A = \{3, 0, 7, a\}$, then $n(A) = 4$, since A has four members. The symbol $n(A)$ may be read as "the cardinal number of A," or simply as "n of A."

Example 13 Let $A = \{m, n, p, q, r\}$. Find $n(A)$.

Solution Since A has five members, $n(A) = 5$.

Example 14 Let $B = \{x | x$ is a natural number less than 5 and greater than $1\}$. Find $n(B)$.

Solution First note that $B = \{2, 3, 4\}$, which has three members. Hence $n(B) = 3$.

A set that has no members is said to be *empty* and is called an *empty set*. For example, the set of three-headed humans is empty. It is convenient to assume that all empty sets are equal. Hence we may speak of *the empty set*, which is denoted by $\emptyset$, or by $\{\ \}$. The empty set is also called *the null set*. Since $\emptyset$ has no members, we may write $n(\emptyset) = 0$.

The empty set is considered to be a subset of every set, even of itself. Furthermore the empty set is considered to be a finite set.

Two sets are *disjoint* iff they have no members in common. For example, the sets $\{1, 2, 3\}$ and $\{4, 7, 9\}$ are disjoint. (The set of elements common to those two sets is the empty set!)

When considering sets it is often useful to refer to some general set, called the *universal set*, from which the members of the considered sets are selected. For example, if we are concerned with sets of students, the universal set might be the set of all college students, the set of all high-school students, or even the set of all students. In any discussion the set being used as the universal set is denoted by U.

Example 15 Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $U = \{x | x$ is a natural number less than $10\}$. State three relationships: (a) one between A and U, (b) one between B and U, and (c) one between A and B.

Answers

a. $A \subset U$, because every member of A is a member of U, and U has at least one member that is not in A.
b. $B \subset U$. (Why?)
c. $A = B$, because A and B have the same members.

Example 16

a. List all the subsets of $\{a, b\}$.
b. List all the subsets of $\{a, b, c\}$.
c. If a finite set has n elements, how many subsets are there?

Answers

a. The subsets of $\{a, b\}$ are $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\}$. Remember that $\emptyset$ is a subset of every set, and every set is a subset of itself.
b. The subsets of $\{a, b, c\}$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.
c. First note that in part (a) we found that a set of 2 elements has 4 subsets. Also observe that $4 = 2^2$. Then note that in part (b) we found that a set of 3 elements has 8, or 2^3, subsets. It can be proved that a set of n elements has 2^n subsets.

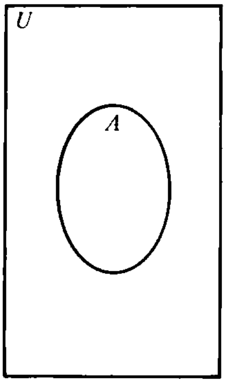


Figure 2

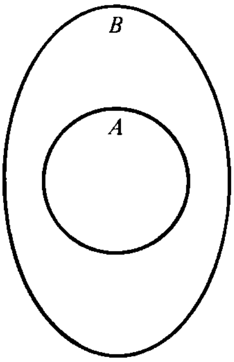


Figure 3

Some relationships between sets can be conveniently represented by plane geometric figures, called *Venn diagrams*. (John Venn was an English logician who lived from 1834 to 1923.) The universal set is usually represented by a rectangle, and other sets are represented by closed curves of various sorts.

Figure 2 indicates that A is a proper subset of the universal set U. Figure 3 indicates that A is a proper subset of B. Figure 4 indicates that A and B are disjoint sets. Figure 5 indicates that A and B have some common members.

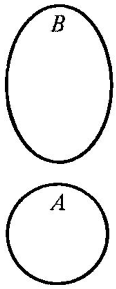


Figure 4

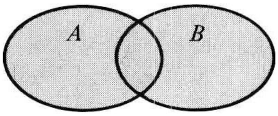


Figure 5

Exercises 1.1

For Exercises 1–6, indicate whether the set is well-defined or ill-defined (not well-defined).

1. The set of students enrolled in this course at your institution.
2. The set of names of the days in any two-week period.
3. The set of successful professors.
4. The set of letters used on this page.
5. The set of natural numbers between 5 and 6.
6. The set of all wealthy people.

For Exercises 7–11, tell whether the set is finite or infinite.

7. The set of Mustang cars manufactured in 1973.
8. The set of all points on a plane.
9. $\{x | x^2 = 9\}$
10. The set of all points exactly two centimers away from a given point.
11. The set of all individual letters used in this book.

For Exercises 12–23, tell whether the statement is true or false. If the statement is false, correct it.

12. $\{5, 6, 7\} = \{7, 6, 5\}$.
13. $\{5, 5, 8\}$ has the same members as $\{5, 8\}$.
14. $3 \subset \{1, 3\}$.
15. $5 \in \{5\}$.
16. $7 \notin \{w | w \text{ is a factor of } 350\}$.
17. The sets $\{4, 6, 8\}$ and $\{x, y, u\}$ are equal because they have the same cardinality.
18. Two infinite sets are not necessarily equal, but it is possible that there may be a one-to-one correspondence between them.

19. The sets $\{2, 3, 4, 5, \ldots\}$ and $\{4, 6, 8, 10, \ldots\}$ are equivalent because there exists a one-to-one correspondence between them.

20. $\{1, 7\} = \{\{1, 7\}\}$. *Note:* The set $\{\{1, 7\}\}$ has only one member.

21. $\emptyset = \{x \mid x + 1 = 1\}$.

22. $\emptyset \not\subset \{6, 8\}$.

23. For any sets A and B, if $A \subset B$, then $B \subset A$.

For Exercises 24–26, express the set in roster notation.

24. The set whose members are 7, 8, and 15.

25. $\{x \mid x + 5 = -3\}$

26. $\{2y \mid 3y - 2 = 6\}$

For Exercises 27–31, express the set in set-builder notation.

27. $\{1, 2, 3, 4, 5\}$

28. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$

29. $\{1, 4, 9, 16, 25, 36\}$

30. The set of natural numbers that are multiples of 3 but less than 22.

31. The set of fractions whose numerators are x and whose denominators are $2x - 1$, where $x \in \{5, 3, 1\}$.

For Exercises 32–34, list all the subsets of the set.

32. $\{n, m\}$

33. $\{\{n, m\}, r\}$

34. $\{1, 7, 8\}$

35. If $3 \in X$, $4 \in Y$, $X \subset Y$, $Y \subset R$, and $R = \{3, 4, 5, 6\}$, then what are the possibilities for the elements of X, and for the elements of Y?

36. Give two examples of three sets such that each set is disjoint from the other two sets.

37. Give two examples of three sets each having exactly one element in common with the other two sets.

38. Give an example of three sets that have the same cardinal number but are not equal sets.

39. Draw a set of lines passing through a fixed point so that the angle between any two adjacent lines is 20°. How many lines are there?

40. Draw a Venn diagram for each of the following set relationships.
   a. $A \subset B$ b. $B \subset A$ c. $A \subset U$ and $B \subset U$, where A and B are disjoint.

41. Draw one Venn diagram that indicates the relationships among the following sets: $A = \{1, 2\}$, $B = \{3, 4\}$, and $U = \{1, 2, 3, 4, 5, 6, 7\}$.

42. Draw a Venn diagram to illustrate that if $A \subset B$ and $B \subset C$, then $A \subset C$.

1.2 Operations on Sets

In the preceding section we discussed some relationships between sets; in this section we will introduce some operations on sets. To fulfill the objectives of this section you should be able to do the following:

a. Distinguish between unary operations and binary operations.
b. Identify the complement of a set A with respect to the universal set, and illustrate the complement by a Venn diagram.
c. Identify the elements in the union of two sets and illustrate the union by a Venn diagram.
d. Identify the elements in the intersection of two sets and illustrate the intersection by a Venn diagram.
e. Simplify set expressions involving more than one operation on at least two sets.

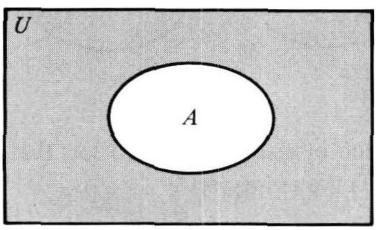
A *unary operation* is an operation performed on a single object. For example, the operation of forming the negative of a real number is a unary operation. Forming the complement of a set (with respect to a given universal set) is also a unary operation.

Given a universal set U, the *complement* of a set A with respect to U is the set of all elements that are in U but not in A. The complement of A is denoted by A'. Thus:

$$A' = \{x | x \in U \text{ and } x \notin A\}$$

In Figure 6, A' is indicated by the shaded region.

Figure 6



Example 17 Let A be the set of vowels: a, e, i, o, and u. And let U be the set of the 26 letters in the alphabet. Find A'.

Solution A' consists of the 21 letters that are not vowels. Thus,

$$A' = \{x | x \text{ is a consonant}\}.$$

A *binary operation* is an operation performed on two objects. For example, the arithmetic operation of multiplication is a binary operation. There are two basic binary operations on sets: union and intersection.

The *union* of two sets A and B is the set consisting of all elements that are members of A, or of B, or of both A and B. The union of A and B is denoted by $A \cup B$, which is read as "A union B." Thus:

$$A \cup B = \{x | x \in A, \text{ or } x \in B, \text{ or } x \in A \text{ and } x \in B\}$$