# Regular and Chaotic Motions in Dynamic Systems

Edited by

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# PREFACE

The fifth International School of Mathematical Physics was held at the Ettore Majorana Centro della Culture Scientifica, Erice, Sicily, 2 to 14 July 1983. The present volume collects lecture notes on the session which was devoted to Regular and Chaotic Motions in Dynamical Systems.

The School was a NATO Advanced Study Institute sponsored by the Italian Ministry of Public Education, the Italian Ministry of Scientific and Technological Research and the Regional Sicilian Government.

Many of the fundamental problems of this subject go back to Poincaré and have been recognized in recent years as being of basic importance in a variety of physical contexts: stability of orbits in accelerators, and in plasma and galactic dynamics, occurrence of chaotic motions in the excitations of solids, etc. This period of intense interest on the part of physicists followed nearly a half a century of neglect in which research in the subject was almost entirely carried out by mathematicians. It is an indication of the difficulty of some of the problems involved that even after a century we do not have anything like a satisfactory solution.

The lectures at the school offered a survey of the present state of the theory of dynamical systems with emphasis on the fundamental mathematical problems involved. We hope that the present volume of proceedings will be useful to a wide circle of readers who may wish to study the fundamentals and go on to research in the subject. With this in mind we have included a selected bibliography of books and reviews which the participants found helpful as well as a brief bibliography for four seminars which were held in addition to the main lecture series.

There were sixty-one participants from sixteen countries.

G. Velo and
A.S. Wightman
Directors of the School

CONTENTS

# REGULAR AND CHAOTIC MOTIONS IN DYNAMICAL SYSTEMS

## INTRODUCTION TO THE PROBLEMS

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The purpose of this introduction is twofold; first, to sketch the origin of some of the problems that will be discussed in detail later, and, second, to introduce some of the concepts which will be used. In a subject like analytical mechanics, with such a long history and such hard problems, a little sense of history is both enlightening and consoling.

A dynamical system is loosely specified as a system with a state at time t given by a point x(t) lying in a phase space, M, and a law of evolution given by an ordinary differential equation (= ODE)

$$\frac{dx}{dt}(t) = v(x(t)) \qquad (1)$$

Here v is a vector field on the phase space M. M is customarily assumed to be a differentiable manifold such as an open set in n-dimensional Euclidean space. Alternatively, one can consider the dynamical system specified by its set of possible histories, the set of mappings, $t \mapsto x(t)$, of some time interval $a < t < b$ into the phase space satisfying the ODE (1). When $-\infty < t < \infty$, the solutions are said to define a flow; when $0 \leq t < \infty$ a semi-flow.

A discrete dynamical system is one in which the time takes integer values. Then the dynamics is given by the iterates of a mapping of the phase space M into itself. If t runs over all the integers, $\mathbb{Z}$, one sometimes speaks of a cascade; if over the positive integers, $\mathbb{Z}_+$, of a semi-cascade. Although the extension of these definitions to infinite dimensional M is of obvious physical interest (fluid dynamics!), in what follows, for lack of time, attention will be mainly confined to the finite dimensional case.

Poincaré's Bequest

The analysis of dynamical systems (= analytical mechanics = classical mechanics = rational mechanics) is one of the oldest parts of physics, but, in a sense, the modern period begins with Poincaré. It is notorious that the physicists of most of the twentieth century had little appreciation of Poincaré's work. Nevertheless, it is his outlook which dominates the field today. To appreciate this, it helps to have been brought up, as I was, on a really old-fashioned version of the subject, say that in E.T. Whittaker's A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. That is a remarkable book, which has some coverage of Poincare's technical results but scarcely a word about his general point of view. Nearly a hundred years later, we find our thinking completely dominated by Poincare's geometric attitude, whether we prefer it in the super-Smalean version of R. Abraham and J. Marsden's Foundations of Mechanics or the proletarian version of V. Arnold's Classical Mechanics.

What then did Poincaré do to exert all this influence? Here is a little list - far from complete.

1) Qualitative Dynamics
   Generic behavior of flows as a whole, the classification of phase portraits.

2) Ergodic Theory
   Probabilistic notions, recurrence theorem.

3) Existence of Periodic Orbits; Detailed Analysis of the Structure of a Flow Near a Periodic Orbit.

4) Bifurcation Theory
   General ideas for systematic theory; detailed study of rotating fluid with gravitational attraction.

First, I will comment briefly on 2). It sounds somewhat anachronistic to call Poincaré a pioneer of ergodic theory but there is a sense in which it is true. In that sense, the first theorem of ergodic theory was the invariance of the Liouville measure while the second was Poincaré's Recurrence Theorem. By the invariance of the Liouville measure. I refer to the fact that

$$dq_1 \ldots dq_n \quad dp_1 \ldots dp_n$$

defines a measure on 2n-dimensional phase space invariant under the flow defined by a Hamiltonian system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots n$$

In modern language, the recurrence theorem can be stated as follows

Theorem

Let T be a mapping of a phase space M into itself which preserves a measure $\mu$ on M:

$$\mu(X) = \mu(T^{-1}X) \quad \text{for any measurable subset } X \text{ of } M$$

Suppose $\mu$ is finite i.e.

$$\mu(M) < \infty$$

Then, if A is any measurable subset of M, almost every point x of A returns to A infinitely often i.e. for an infinite set of positive integers, n, $T^n x \in A$.

Poincaré emphasized that his proof required only the finiteness and invariance of his measure, although the argument used the language of the theory of incompressible fluids. He had already gone far in the direction of generality in these matters by introducing the general notion of integral invariants. These are invariant integrals of differential forms over subsets of M.

Incidentally, for those who may wish to read the original, I should note that Poincaré did not call this result a recurrence theorem; he referred to it as stabilité à la Poisson. You can find it, along with a magistral exposition of his theory of integral invariants in his Prize Memoir which won (21 January 1889) the Prize offered by King Oscar II of Sweden. It is published in Acta Math 13 (1890) 1-270.

It is interesting to compare this stunningly general result with what was going on in physics at that time. Maxwell and Boltzmann had constructed statistical models of gases leading to quantitative predictions of thermodynamic phenomena, and Boltzmann had published a proof of the so-called H-Theorem giving a mechanical interpretation of the increase of entropy in accord with the Second Law of Thermodynamics. Boltzmann's proof was greeted with skepticism because of the Recurrence Theorem and the invariance under time inversion of the usual Hamiltonian models. Both Maxwell and Boltzmann made independent efforts to justify statistical procedures on the basis of what Boltzmann called the Ergodic Hypothesis:

the trajectory of a Hamiltonian system in phase space passes through every point of its surface of constant energy. Poincaré thought it very unlikely that a single trajectory could fill a whole surface of constant energy.[1] (A theorem to this effect was proved much later by A. Rosenthal and M. Plancherel.[2]) He immediately replaced it by the more plausible assumption that every orbit is dense, a property later called the Quasi-Ergodic Hypothesis by the Ehrenfests, in their well-known article in the Mathematical Encyclopedia.[3] Even in the 1890's Poincaré knew too much about the behavior of orbits in concrete dynamical systems to believe in the general validity of the Ouasi-Ergodic Hypothesis. He pointed out that in the restricted problem of three bodies (interacting with gravitational attractions) there are orbits not dense on the surfaces of constant values of the integrals of motion. His general attitude was summarized[1]

> "Il est possible et même vraisemblable que le postulat de Maxwell est vrai pour certains systèmes et faux pour d'autres, sans qu'on ait aucun moyen certain de discerner les uns des autres."

As will be discussed in the following, we now have some means of distinguishing ergodic from non-ergodic systems and the first part of the sentence has turned out to be exactly right.

After the Ehrenfests most theoretical physicists stayed away from the problem. The only exception I know was Enrico Fermi.[4] In 1923, he extended a theorem of Poincaré to show that for Hamiltonian systems of n degrees of freedom with $n > 2$ satisfying certain conditions of genericity, there could exist no smooth hypersurface of dimension $2n-1$ invariant under the flow except for the surfaces of constant energy. He then applied his result to prove that such systems would have to be ergodic because, if there existed an open subset of the phase space invariant under the flow, its boundary would be a hypersurface to which the preceding theorem would apply. This argument assumes that the only subsets which have to be considered are those with smooth boundaries, an assumption which is now known to fail in general as a result of the KAM theorem. (See Giovanni Gallavotti's lectures.) Fermi's argument would have meant that generically there could be no nontrivial invariant decomposition of the energy surface. (We now know there are large interesting classes of Hamiltonian systems for which the orbits are dense on the energy surface and large interesting classes for which they are not. In the latter case, the boundary of an invariant subset is rough, violating Fermi's assumption.)

Now I turn to 3), the analysis of a flow near a periodic orbit. It was here that Poincaré uncovered many of the problems that have evolved into the main subjects of the lectures that follow.

When Poincaré began his work, the result which we now know as the rectification theorem was standard knowledge In a somewhat modernized form, it is as follows.

Theorem

If v is a continuously differentiable vector field defined in a neighborhood of a point $x_0$ , where $v(x_0) \neq 0$, then in some sub-neighborhood of $x_0$, there exists a continuously differentiable change of coordinates $x \to y = f(x)$ such that the differential equation

$$\frac{dx}{dt} = v(x)$$

is reduced to the form

$$\frac{dy_1}{dt} = 1, \quad \frac{dy_2}{dt} = \frac{dy_3}{dt} = \cdots \frac{dy_n}{dt} = 0$$

In pictures

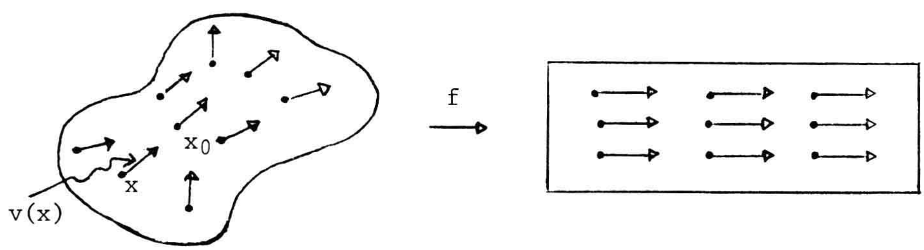


Figure 1. The vector field v in a neighborhood of x is rectified by the mapping f.

In the coordinates y, $y_1$ increases proportionally to t, and $y_2,\ldots y_n$ are integrals of motion. The Rectification Theorem asserts that a smooth flow is very simple in a neighborhood of a non-singular point i.e. a point where the vector field is non-vanishing. If a flow is this simple locally, it is natural to ask why one cannot use the n-1 local integrals of motion to obtain a simple global labeling of orbits. The answer is complicated in general. Sometimes, one can and then one has an integrable system. Sometimes, one can do it over a large region but runs into difficulties when trying to extend to the whole phase space. The extension may be impossible because the vector field v has a singular point at some $x_0$ i.e. $v(x_0) = 0$. Alternatively, it may be impossible because, as one approaches some subset , the neighborhoods that the rectification theorem provides get smaller and smaller. For any finite time the local rectifications could be patched together but in the limit of infinite time, it would be impossible.

However, the labeling of orbits by $y_1 \ldots y_n$ may fail for a simpler reason based on the phenomenon of recurrence, which we know to be very general from Poincaré's Recurrence Theorem. If the orbit of a particle goes away but returns to a neighborhood of its starting point, it may not be possible to patch together the coordinate systems so that they give the orbit the same values of $y_2 \ldots y_n$ as when it started out. This phenomenon does not necessarily involve any lack of smoothness or singularity of the vector field; it is a matter of global geometry of phase space. (You will hear this eloquently expounded by Trubowitz in his lectures on integrable systems.)

In the context of the Newtonian N-body problem (N bodies of masses $m_1 \ldots m_N$ interacting by gravitational attraction) the question of the existence of global integrals of motion in addition to the ten well-known integrals

Energy $$E = \sum_{j=1}^{N} \frac{1}{2} m_j \left(\frac{d\vec{x}_j}{dt}\right)^2 + \sum_{j<k} \frac{Gm_j m_k}{|\vec{x}_m - \vec{x}_k|}$$

Momentum $$\vec{P} = \sum_{j=1}^{N} m_j \frac{d\vec{x}_j}{dt}$$

Angular Momentum $$\vec{J} = \sum_{j=1}^{N} [\vec{x}_j \times m_j \frac{d\vec{x}_j}{dt}]$$

Center of Mass $$M\vec{X} - \vec{P}t \quad \text{where} \quad \vec{X} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{x}_j \, , \quad M = \sum_{j=1}^{N} m_j$$

was a famous nineteenth century problem. It was a result of Bruns that no additional independent integrals exist depending algebraically on the coordinates momenta and time. Poincaré generalized this result; he showed that algebraic could be replaed by holomorphic - this was another result of his Prize Memoir of 1889, the result which Fermi extended in the work referred to above.

Clearly, to go farther one has to understand the nature of a flow near a singular point. (This is also the simplest case of a periodic orbit: all positive real numbers are periods.)

Poincaré's first results in this direction were contained in his thesis (1879) although the main emphasis there is on other matters.[5] The thesis, in fact, is mainly about first order partial differential equations (= PDEs), and Poincaré was principally concerned with completing the Cauchy-Kowalewski theory for those equations, so that it covered certain cases which previous authors had not treated. However, in the course of this investigation, he discussed the ODEs for a bicharacteristic strip in a neighborhood of an equilibrium point.

Recall that if the PDE is

$$F(x_1 \dots x_n, u, p_1 \dots p_n) = 0$$

where $u$ is the unknown function of $x_1 \dots x_n$ and $p_1 = \partial u/\partial x_1 \dots p_n = \partial u/\partial x_n$, then associated with it is a system of ODEs

$$\frac{dx_j}{dt} = \frac{\partial F}{\partial p_j}(x_1 \dots x_n, u, p_1 \dots p_n) \qquad j = 1, \dots n$$

$$\frac{dp_j}{dt} = -\left[\frac{\partial F}{\partial x_j}(x_1 \dots x_n, u, p_1 \dots p_n) + p_j \frac{\partial F}{\partial u}\right] \qquad j = 1, \dots n$$

$$\frac{du}{dt} = \sum_{j=1}^{n} p_j \frac{\partial F}{\partial p_j} \qquad (2)$$

in $\mathbb{R}^{2n+1}$, the ODEs for a bicharacteristic strip.[6] When the right-hand side of (2) vanishes at the initial point, it is called an equilibrium point. More generally, for the ODE (1), one calls $x_0$ a <u>singular point</u> of the vector field $v$ if $v(x_0) = 0$. For the general purposes of the present discussion, the special features of the system (2) for the bicharacteristics are not significant (that was also the case in Poincaré's thesis) so I will continue the discussion in terms of the general equation.

For convenience, assume that the singular point $x_0$ is at the origin of coordinates. There is then a special case in which the qualitative behavior of the solutions is determined by the eigenvalues of A. For a general differentiable vector field with a singular point at 0, the Taylor expansion of $v$ provides a linear transformation with $A = (\nabla v)/_0$ and nonlinear correction terms. Poincaré posed the problem: Under what conditions will a change of coordinates $x \mapsto \phi(x)$ leaving the origin fixed, $\phi(0) = 0$, reduce a vector field to its linear part in a neighborhood of zero:

$$v(\phi(x)) = \phi(Ax)$$

Poincaré assumed $v$ analytic, solved the problem when $\phi$ is regarded as a formal power series, and then gave sufficient conditions that the formal power series converge. The key to the first step is the idea of resonance:

*Definition* Let $\lambda_1 \ldots \lambda_n$ be the eigenvalues of the matrix A, and $x^m = \sum_{j=1}^{n} x_j^{m_j}$ be a monomial occurring in the Taylor expansion of v. Then $x^m$ is *resonant* if for some s, $1 \leq s \leq n$

$$\lambda_s = \sum_{j=1}^{n} \lambda_j m_j \quad \text{with} \quad \sum_{j=1}^{n} m_j \geq 2$$

In the recursive procedure which Poincaré found for the determination of the coefficients of the power series for $\phi$ the quantities

$$\lambda_s - \sum_{j=1}^{n} \lambda_j m_j$$

occur in denominators. For a resonant term the procedure fails. Even if there are no resonant terms there may be trouble at the next stage if the denominators get small - that is the famous *problem of small divisors* - and Poincaré's sufficient condition for the convergence of the power series for $\phi$ yields uniform boundedness away from zero of the denominators. Later on, Dulac modified Poincare's procedure so as to give a solution of the *normal form problem*: find a formal power series which transforms v(x) to the linear term Ax plus resonant terms.[7] All this leaves open what happens to the flow in the presence of resonances or where there are no resonances but Poincaré's sufficient condition for convergence is not satisfied. Poincaré's thesis uncovered the hard nut of problems which are still with us. For further details of the Poincaré-Dulac theorems, see Chapter V of Arnold's book *Chapitres Supplémentaires...* .[8]

The previous analysis treats orbits in a neighborhood of a singular point by finding a coordinate system in which the vector field on the right-hand side of the ODE is reduced to normal form, which will be linear in favorable cases. When the procedure goes through, the linear approximation describes the exact behavior in the new coordinate system. This analysis is not directly applicable to periodic orbits of strictly positive period, $T > 0$, but Poincaré developed a different approach which makes an analogous analysis possible. Given a periodic orbit, one picks a point on it and chooses an n-1 dimensional manifold passing through $x_0$ and not tangent to the periodic orbit i.e. it is *transversal* to the orbit. It is sometimes called a *Poincaré section*. See Figure 2. Then through a point x on the section near x there passes an orbit which will come back and hit the section near x, but not necessarily at x because not every orbit near $x_0$ need be periodic.

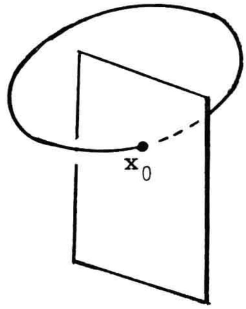


Figure 2 The Poincaré Section is transversal to the periodic orbit at $x_0$. The Poincaré mapping $\phi$ carries the section into itself.

Thus the flow defines a mapping, $\phi$, of a neighborhood of $x_0$ on the section into itself, the "once around" mapping, usually called the Poincaré mapping associated with the periodic orbit and $x_0$ and the Poincaré section. The point $x_0$ is a fixed point of the Poincaré mapping

$$\phi(x_0) = x_0$$

An expansion of $\phi(x)$ in Taylor series about $x_0$ has no constant term and the linear term $[(\nabla\phi)(x_0)](x-x_0)$ can be regarded as an analogue of the linear approximation Ax to a vector field near an equilibrium point. One can study the asymptotic behavior of orbits near the periodic orbit by studying the iterates $\phi^n x$ under the Poincaré mapping. Notice that the first derivative of $\phi^n$ evaluated at $x_0$ is just the nth power of the matrix $(\nabla\phi)(x_0)$ so the asymptotics in linear approximation can be read off from this matrix. Afterwards, one will put the nonlinear terms back in and see what qualitative features survive.

Up to this point, everything that has been said applies to a general ODE. Now, I turn to the more special results which hold if the ODE is Hamiltonian. Here we have the remarkable fact that the Poincaré mapping $\phi$ is always symplectic i.e. on the surface of section it preserves the symplectic structure which that surface of section inherits from the general symplectic structure of the phase space. This has the consequence that the eigenvalues of $(\nabla\phi)(x_0)$ always appear in quadruples $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$. If $\lambda$ is on the unit circle, the traditional terminology calls it elliptic while if $|\lambda| \neq 1$ it is called hyperbolic. The associated patterns of flow are strikingly different, as one sees from the two dimensional example illustrated in Figure 3. This two-dimensional example becomes relevant in a Hamiltonian system of two degrees of freedom where the energy surface, $E(q_1 q_2 p_1 p_2)$ = const, is three-dimensional and a Poincaré section is two-dimensional. The invariant curves