Fundamentals of Actuarial Mathematics

Second Edition

S. David Promislow

$$\frac{d}{dt}\,{}_t\bar{V} = \delta({}_t\bar{V}) + \pi(t) - \mu_x(t)(b(t) - {}_t\bar{V})$$

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Preface

Several factors motivated the writing of this book. After teaching undergraduate actuarial courses for many years, it became clear that there was a definite need for more instructional material in this area. In most undergraduate courses, students who had problems reading a particular text could go to the library and find dozens of other references which might assist them. By comparison there were very few resources dealing with actuarial mathematics.

In addition, there was need for a book which would give full recognition to modern computing methods and techniques. Many existing books still emphasize material which was developed in a time when calculations were done by hand. At present, basic actuarial calculations are easily done using computer spreadsheets, and I felt it was time for a text which would develop the ideas and methods with this in mind.

The book covers two fundamental topics in actuarial mathematics. These are life contingencies and risk theory, including the basics of ruin theory.

The modern approach towards life contingencies is through a stochastic model, as opposed to the older deterministic viewpoint. I certainly agree that mastering the stochastic model is the desirable end. However, my classroom experience has convinced me that this is not the right place to begin the instruction. I find that students are much better able to learn the new ideas, the new notation, the new ways of thinking involved in this subject, when done first in the simplest possible setting, namely a deterministic discrete model, and I have followed this approach in this book. After the main ideas are presented in this fashion, continuous models are introduced. In Part II of the book, the full stochastic model can then be dealt with in reasonably quick fashion.

The book covers a great deal of the material on the modeling exams of the Society of Actuaries and the Casualty Actuarial Society. A major audience for the book will be students preparing for these exams. The order of topics, however, provides a degree of flexibility, so that the book can be of interest to different readers. Part I of the book will serve the needs of those who want only an introduction to the subject, without necessarily specializing in it. The only mathematical background required for this material is some elementary linear algebra and probability theory, and, beginning in Chapter 8, some basic calculus.

A more advanced knowledge of probability theory is needed from Chapter 13 onward. All of this material is summarized in Appendix A. Basic concepts of stochastic processes are used in Part III of the book, which deals with the collective risk model. These are developed in the text in Chapters 18 and 19.

For the most part, we do not include statistical aspects of the subject, unlike for example Klugman *et al.* (2008). Rather, the emphasis is on methods of using the information that the statistician would produce. No prior knowledge of statistical inference, as opposed to probability theory, is required.

A usual prerequisite for this type of material is a course in the theory of interest. Although this may be useful, it is not strictly required. All the interest theory that is needed is presented as a particular case of the general deterministic actuarial model in Chapter 2.

A major source of difficulty for many students in learning actuarial mathematics is to master the rather complex system of actuarial notation. We have introduced some notational innovations, which tie in well with modern calculation procedures as well as allowing us to greatly simplify the notation that is required. We have, however, included all the standard notation in separate sections, at the end of the relevant chapters, which can be read by those readers who desire this material.

The book is intended to cover the material at a basic level and is not as encyclopedic as a work like Bowers *et al.* (1997). To meet this goal, and to keep the length reasonable, we have necessarily had to omit certain important topics. The most notable of these is stochastic interest rates. There is a brief discussion of this idea, but for the most part interest rates are taken as deterministic. There is more of an emphasis on life insurance and annuities as opposed to casualty insurance. Some important casualty topics, such as loss reserving, are not covered here.

Keeping in mind the nature of the book and its intended audience, we have avoided excessive mathematical rigor. Nonetheless, careful proofs are given in all cases where these are thought to be accessible to the typical senior undergraduate mathematics student. For the few proofs not given in their entirety, mainly those involving continuous-time stochastic processes, we have tried at least to provide some motivation and intuitive reasoning for the results.

Exercises appear at the end of each chapter. In Parts I and II these are divided up into different types. Type A exercises generally are those which involve direct calculation from the formulas in the book. Type B involve problems where more thought is involved. Derivations and problems which involve symbols rather than numeric calculation are normally included in Type B problems. A third type is spreadsheet exercises which themselves are divided into two subtypes. The first of these ask the reader to solve problems using a spreadsheet. Detailed descriptions of applicable Microsoft Excel® spreadsheets are given at the end of the relevant chapters. Readers of course are free to modify these or construct their own. The second subtype does not ask specific questions but instead asks the reader to modify the given spreadsheets to handle additional tasks. Answers to most of the calculation-type exercises appear at the end of the book.

Sections marked with an asterisk * deal with more advanced material, or with special topics that are not used elsewhere in the book. They can be omitted on first reading. The exercises dealing with such sections are likewise marked with *.

The material in the book comprises approximately three semesters of work in the typical North American university. A rough guide would be to do Chapters 1–8 in the first semester, Chapters 9–16 in the second semester, and Chapters 17–23 in the third. Part III is for the most part independent of Parts I and II. A major exception is Chapter 23, which generalizes material in Chapter 11, and can be read immediately after that chapter, for the reader with a basic knowledge of Markov chains, as presented in Chapter 19. Another exception is Section 20.4.1 which alludes to previous material. Chapters 7 (except for Section 7.3.1), 9 and 12 deal with topics that are important in applications, but which are not used in other parts of the book. They could be omitted without loss of continuity.

Changes in the second edition

There are several additions and changes for the second edition. The most important of these of these are the inclusion of three new chapters, and substantial modifications to a few others.

A chapter on *credibility theory,* has been added. This is a major actuarial topic which was not addressed in the first edition. While the emphasis of the book is still on the life and pension side of actuarial science, this chapter provides additional material for those whose main interest is in casualty insurance.

A chapter was added on *risk assessment,* another major subject area which received only minimal coverage in the first edition. The theme here is the comparison and measurement of risk in random alternatives, and the chapter introduces such topics as utility theory, a stochastic ordering method, and risk measures, with a concentration on VaR and TailVaR.

The subject of *multi-state* models, has proved to be an effective way of unifying much of actuarial theory. Some aspects of the discrete model were included in the first edition as an application of Markov chain theory. This material has been extended and combined with the continuous time model, to form a new chapter on this topic.

Chapter 10 has been extended to include situations involving a a duration that runs from a death of an individual rather than from time zero. This provides additional techniques, and equips the reader to handle a greater variety of multiple-life contracts. The chapter also includes a section outlining applications to credit risk in annuities.

In the first edition, the multiple-decrement theory was contained in two chapters, the classical model in Chapter 11 and a more general treatment in Chapter 16. In this edition, much of the Chapter 16 material has been rewritten and moved back to Chapter 11, so that this earlier chapter now contains a more complete exposition of the subject.

Other changes include the following:

- In Chapter 2 there is some additional material dealing with forward prices and term structure for bonds.
- A section has been added to Chapter 6, outlining the provisions of some modern types of contracts such as universal life and variable annuities.
- In Chapter 9 on select mortality, there is a new section illustrating how projections in annuity tables fit into the select framework.
- The method of presentation of some of the preliminary material has been changed, and time diagrams are introduced as a visual aid for depicting insurance and annuity contracts.
- The spreadsheets covering the early chapters have been modified to improve efficiency of use.
- Additional examples and exercises have been added to several chapters.

This book includes an accompanying website. Please visit www.wiley.com/go/actuarial for more information.

Acknowledgements

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Notation index

The following is a list of the major symbols which are used in the book. For the most part, the first page they appear on is listed. An exception is the notation in Appendix A, where the first appearance in that chapter is noted. This list excludes that part of the standard actuarial notation which is not used in the main body of the text. The latter can be found in the approriate sections of Chapters 2–6, 8 and 10 entitled 'Standard Notation and Terminology'.

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Chapter 21

Chapter 22

Chapter 23

Appendix A

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