Series on Partial Differential Equations and Applications – Vol. 2

Yihong Du

Order Structure and Topological Methods in Nonlinear Partial Differential Equations

Vol 1 Maximum Principles and Applications

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Order Structure and Topological Methods in Nonlinear Partial Differential Equations

Vol 1 Maximum Principles and Applications

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Vol. 2 Order Structure and Topological Methods in Nonlinear Partial Differential Equations
Vol. 1 – Maximum Principles and Applications
by Yihong Du (Univ. of New England, Australia & Qufu Normal Univ, China)

Preface

This is volume one of a two volume series. The intention is to provide a reference book for researchers in nonlinear partial differential equations and nonlinear functional analysis, especially for postgraduate students who want to be led to some of the current research topics. It could be used as a textbook for postgraduate students, either in formal classes or in working seminars.

In these two volumes, we attempt to use order structure as a thread to introduce the various versions of the maximum principles, the fixed point index theory, and the relevant part of critical point theory and Conley index theory. The emphasize is on their applications, and we try to demonstrate the usefulness of these tools by choosing applications to problems in partial differential equations that are of considerable concern of current research.

An important work in this direction is H. Amann's classical review article (SIAM Rev. 18 (1976), 620-709), which discussed the combination of order structure and fixed point index theory and its applications to various problems of nonlinear partial differential equations. Much progress has been made since this article. The fixed point index theory has been further developed and found important new applications in partial differential equations. Moreover, the order structure has since been successfully combined with critical point theory and Conley index theory to study various nonlinear partial differential equation problems. Furthermore, the classical maximum principle in partial differential equations has found new applications in several important problems. All these are scattered in research articles published in various professional journals, and most of them are still active topics of current research.

It is our hope that through these two volumes, we can present the reader in a somewhat systematic way some of the new progresses in these topics. As the title suggests, volume 1 mainly considers the maximum principles and their various applications in some of the current research topics. The topological methods will be discussed in volume 2.

There are 7 chapters and an appendix in this volume 1. In chapter 1, we use the Krein-Rutman theorem to derive several well-known properties of the principal eigenvalues, we then use these in chapter 2 to characterize the maximum principle. We briefly discuss the moving plane method in chapter 3. Existence results are not discussed until chapter 4, where we consider the methods of upper and lower solutions, also known as super and sub-solution methods. The weak theory here is based on the theory of monotone operators, whose basic result is recalled without proof. With these preparations, existence results can be considered in the later chapters. In chapter 5, the basic logistic model is discussed, where various comparison arguments, the upper and lower solution methods, together with a variety of elliptic estimates are used. Chapter 6 gives an introduction of some basic boundary blow-up problems. The last chapter considers again various symmetry properties of elliptic problems, where apart from the moving plane methods, other techniques are also used. In the appendix, we include a brief review of the classical elliptic theory for second order partial differential equations. Since this basic theory may take a long time for the beginners to master, we feel it might be practical for those readers such as postgraduate students to initially accept the relevant basic results in this theory and continue with their study of some current research topics.

Some of the material is chosen to be included here for its usefulness, such as the various versions of the maximum principles, and the upper and lower solution methods. In such a case, we have tried to make the results as general as possible, provided that not too much complication of the presentation is caused. Some of the material is included here in order to introduce useful techniques and to lead the reader to some of the current research problems. In this situation, we usually put clarity in front of generality. The material presented and the references quoted here are mainly based on the author's taste and familiarity, which inevitably are biased with many important topics and references not included here. I apologize if these omissions inadvertently offend anyone.

In volume 2, we will discuss some developments of the fixed point index theory (mainly due to E.N. Dancer), and their applications to various problems, in particular to several population models. We will also discuss the part of critical point theory and Conley index theory that can be combined with order structure to provide better applications. Some of the material in volume 1 here provides necessary preparation for volume 2.

It is my great pleasure to thank all those who helped in one way or another in the writing of this first volume. In particular, I would like to express my deep thanks to Professor Norman Dancer for guiding me into nonlinear analysis, and for the constant help and encouragements. My sincere thanks to Professor Dajun Guo for taking me into nonlinear functional analysis, and to Professor Xingbin Pan for encouraging me to write this book. I'm grateful to my colleagues at the University of New England who freed me from teaching duties in the first half of 2005; that helped immensely in getting this belated volume ready before the end of the year. Part of the material here was presented at Qufu Normal University at a workshop in 2004, and my thanks go to the colleagues there for the help and support. Over the years, I have benefitted greatly from working with my collaborators. My sincere thanks to all of them, in particular, Florica Cîrstea, Zongming Guo, Shujie Li, Lishan Liu, Li Ma, Tiancheng Ouyang, Shusen Yan, Feng Zhou, and my former PhD students and friends Qingguang Huang and Wei Dong, whose joint papers with me are used in this volume. Several friends helped me proof reading various parts of this volume, and I would like to thank in particular Florica Cîrsta, Xing Liang, Rui Peng and Shusen Yan for their efforts in helping me reducing the mistakes. It is my responsibility for any remaining mistakes, and corrections from the readers are very much appreciated. My sincere thanks also go to the editors of World Scientific Publishing, especially Ms Zhang Ji, for all the help and advices. Finally I thank my family for the understanding and support during the writing of this book.

Yihong Du
September, 2005, Armidale

Contents

Chapter 1

Krein-Rutman Theorem and the Principal Eigenvalue

The Krein-Rutman theorem plays a very important role in nonlinear partial differential equations, as it provides the abstract basis for the proof of the existence of various principal eigenvalues, which in turn are crucial in bifurcation theory, in topological degree calculations, and in stability analysis of solutions to elliptic equations as steady-state of the corresponding parabolic equations. In this chapter, we first recall the well-known Krein-Rutman theorem and then combine it with the classical maximum principle of elliptic operators to prove the existence of principle eigenvalues for such operators.

Let X be a Banach space. By a *cone* $K \subset X$ we mean a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone K in X induces a *partial ordering* $\leq$ by the rule: $u \leq v$ if and only if $v - u \in K$. A Banach space with such an ordering is usually called a partially ordered Banach space and the cone generating the partial ordering is called the *positive cone* of the space. If $\overline{K-K} = X$, i.e., the set $\{u - v : u, v \in K\}$ is dense in X, then K is called a *total cone*. If $K - K = X$, K is called a *reproducing cone*. If a cone has nonempty interior K^0, then it is called a *solid cone*. Any solid cone has the property that $K - K = X$; in particular, it is total. Indeed, choose $x_0 \in K^0$ and $r > 0$ such that the closed ball $B_r(u_0) := \{u \in X : \|u - u_0\| \leq r\}$ is contained in K. Then for any $u \in X\backslash\{0\}$, $v_0 := u_0 + ru/\|u\| \in K$ and hence $u = (\|u\|/r)(v_0 - u_0) \in K - K$. We write $u > v$ if $u - v \in K \setminus \{0\}$, and $u \gg v$ if $u - v \in K^0$.

Let X^* denote the dual space of X. The set $K^* := \{l \in X^* : l(x) \geq 0 \ \forall x \in K\}$ is called the dual cone of K. It is easily seen that K^* is closed and convex, and $\lambda K^* \subset K^*$ for any $\lambda \geq 0$. However it is not generally true that $K^* \cap (-K^*) = \{0\}$. But if K is total, this last condition is satisfied and hence K^* is a cone in X^*. Indeed, if $l \in K^* \cap (-K^*)$, then for every

$x \in K$, $l(x) \geq 0, -l(x) \geq 0$, and therefore $l(x) = 0$ for all $x \in K$. Since $\overline{K-K} = X$, this implies that $l(x) = 0$ for all $x \in X$, i.e., $l = 0$.

Let Ω be a bounded domain in R^N. It is easily seen that the set of nonnegative functions K in $X = L^p(\Omega)$ is a cone satisfying $K - K = X$. However, it has empty interior. Similarly the set of nonnegative functions in $W^{1,p}(\Omega)$ gives a reproducing cone, and generally the nonnegative functions in $W^{k,p}(\Omega)$ $(k \geq 2, p > 1)$ form a total cone. On the other hand, the nonnegative functions form a solid cone in $C(\overline{\Omega})$ but only form a reproducing cone in $C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. If Ω has C^1 boundary $\partial\Omega$, then it is easy to see that the nonnegative functions in $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ form a solid cone; for example, any function satisfying $u(x) > 0$ in Ω and $D_\nu u(x) < 0$ on $\partial\Omega$ is in the interior of the cone, where ν denotes the outward unit normal of $\partial\Omega$.

Theorem 1.1 (The Krein-Rutman Theorem, [Deimling(1985)] Theorem 19.2 and Ex.12) *Let X be a Banach space, $K \subset X$ a total cone and $T : X \to X$ a compact linear operator that is positive (i.e., $T(K) \subset K$) with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus \{0\}$: $Tu = r(T)u$. Moreover, $r(T^*) = r(T)$ is an eigenvalue of T^* with an eigenvector $u^* \in K^*$.*

Let us now use Theorem 1.1 to derive the following useful result.

Theorem 1.2 *Let X be a Banach space, $K \subset X$ a solid cone, $T : X \to X$ a compact linear operator which is strongly positive, i.e., $Tu \gg 0$ if $u > 0$. Then*

(a) $r(T) > 0$, and $r(T)$ is a simple eigenvalue with an eigenvector $v \in K^0$; there is no other eigenvalue with a positive eigenvector.
(b) $|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.

Let us recall that r is a simple eigenvalue of T if there exists $v \neq 0$ such that $Tv = rv$ and $(rI - T)^n w = 0$ for some $n \geq 1$ implies $w \in span\{v\}$.

Proof. **Step 1**: *There exists $v_0 > 0$ such that $Tv_0 = r(T)v_0$.*

Fix $u \in K^0$. Then $\alpha Tu \geq u$ for some $\alpha > 0$, and we can find $\sigma > 0$ such that $B_\sigma(u) \subset K$. It follows that $w \leq (\sigma)^{-1}\|w\|u$ for any $w \in X$. Let $S = \alpha T$. Then

$$u \leq S^n u \leq \sigma^{-1}\|S^n u\|u \leq \sigma^{-1}\|S^n\|\|u\|u, \ \forall n \geq 1.$$

Hence

$$\|S^n\| \geq \sigma/\|u\| \text{ and } r(S) = \lim_{n\to\infty} \|S^n\|^{1/n} > 0.$$

By Theorem 1.1, $r(S)$ is an eigenvalue of S corresponding to a positive eigenvector $v_0 \in K \setminus \{0\}$. Clearly $r(T) = r(S)/\alpha > 0$ and $Tv_0 = r(T)v_0$.

Step 2: *To prove that $r(T)$ is simple, we show a more general conclusion: If $r > 0$ and $Tv = rv$ for some $v > 0$, then r is a simple eigenvalue of T.*

Let us first show that $(rI - T)w = 0$ implies $w \in span\{v\}$. Suppose $Tw = rw$ with $w \neq 0$. Then $T(v \pm tw) = r(v \pm tw)$ for all $t > 0$. Since T is strongly positive, $v \in K^0$ and the above identity implies $v \pm tw \notin \partial K$ unless $v \pm tw = 0$. But $v \pm tw \in K^0$ for small t and this cannot hold for all large t for otherwise $w \in K \cap (-K) = \{0\}$. Therefore there exists $t_0 \neq 0$ such that $v + t_0 w \in \partial K$ and hence $v + t_0 w = 0$. This proves $w \in span\{v\}$.

Let $(rI - T)^2 w = 0$. By what has just been proved, $rw - Tw = t_0 v$ for some $t_0 \in R^1$. If $t_0 \neq 0$, then we may assume $t_0 > 0$ (otherwise change w to $-w$). Since

$$T(v + sw) = r(v + sw) - st_0 v \ll r(v + sw) \text{ for all } s > 0,$$

and $v + sw \in K^0$ for all small $s \geq 0$, we easily deduce $v + sw \in K^0$ for all $s \geq 0$. This implies that $w \in K$, and hence $w = r^{-1}(t_0 v + Tw) \in K^0$. We now have

$$w - tv \in K^0 \text{ for all small } t > 0,$$

but not for all large $t > 0$ as this would imply $v = 0$. Therefore there exists $t_1 > 0$ such that $w - t_1 v \in \partial K$. But then

$$rw - t_0 v - t_1 rv = T(w - t_1 v) \geq 0, \; w - t_1 v \geq r^{-1} t_0 v \gg 0,$$

contradicting $w - t_1 v \in \partial K$. Therefore we must have $t_0 = 0$ and hence $rw - Tw = 0$, $w \in span\{v\}$. This proves that r is a simple eigenvalue.

Step 3: *Next we show that T cannot have two positive eigenvalues $r_1 > r_2$ corresponding to positive eigenvectors:*

$$Tv_1 = r_1 v_1, \; Tv_2 = r_2 v_2.$$

Let $v(t) = v_2 - tv_1, t \geq 0$. Since T is strongly positive, we have $v_1, v_2 \in K^0$. As before we have $v(t) \in K^0$ for small t but not for all large t.

Therefore there exists $t_0 > 0$ such that $v(t_0) \in K$ but $v(t) \notin K$ for $t > t_0$. We now have

$$v_2 - t_0(r_1/r_2)v_1 = r_2^{-1}T(v_2 - t_0v_1) \in K,$$

which implies $r_1 \leq r_2$ due to the maximality of t_0. This contradiction proves step 3.

Step 4: *If $Tw = \lambda w$ with $w \neq 0$ and $\lambda \neq r(T)$, then $|\lambda| < r(T)$.*

If $\lambda > 0$, then by Step 3, $w \notin K$. It follows that $v_0 + tw \in K$ for all small $t > 0$ but not for all large t. Therefore there exists $t_0 > 0$ such that $v_0 + t_0w \in K$ and $v_0 + tw \notin K$ for $t > t_0$. It then follows that $v_0 + t_0(\lambda/r(T))w = r(T)^{-1}T(v_0 + t_0w) \in K$. The maximality of t_0 implies that $\lambda \leq r(T)$ and hence $\lambda < r(T)$.

If $\lambda < 0$, then from $T^2w = \lambda^2 w$ and $T^2 v_0 = r(T)^2 v_0$ and the above argument (applied to T^2) we deduce $\lambda^2 < r(T)^2$ and hence $|\lambda| < r(T)$.

Consider now the case that $\lambda = \sigma + i\tau$ with $\tau \neq 0$. Then necessarily $w = u + iv$ and

$$Tu = \sigma u - \tau v, \; Tv = \tau u + \sigma v. \tag{1.1}$$

We observe that u and v are linearly independent for otherwise we necessarily have $\tau = 0$. Let $X_1 := span\{u, v\}$. Then (1.1) implies that X_1 is an invariant subspace of T. We claim that $K_1 := X_1 \cap K = \{0\}$. Otherwise K_1 is a positive cone in X_1 with nonempty interior, as for any $w \in K_1 \setminus \{0\}$, $Tw \in X_1 \cap K^0 = K_1^0$. We can now apply Step 1 above to T on X_1 to conclude that there exists $r > 0$ and $w_0 \in K_1^0$ such that $Tw_0 = rw_0$. By Steps 2 and 3, we necessarily have $r = r(T)$ and $w_0 \in span\{v_0\}$. In other words, $v_0 \in K_1$ and $v_0 = \alpha u + \beta v$ for some real numbers α and β. But then one can use (1.1) and $Tv_0 = r(T)v_0$ to easily derive $\alpha = \beta = 0$, a contradiction. Therefore $K_1 = \{0\}$.

From $span\{u, v\} \cap K = \{0\}$ we find that the set

$$\Sigma := \{(\xi, \eta) \in R^2 : v_0 + \xi u + \eta v \in K\}$$

is bounded and closed. Since $v_0 \in K^0$, $M := \sup\{\xi^2 + \eta^2 : (\xi, \eta) \in \Sigma\} > 0$ and is achieved at some $(\xi_0, \eta_0) \in \Sigma$. Let $z_0 = v_0 + \xi_0 u + \eta_0 v$. Then $z_0 \in K \setminus \{0\}$ and $Tz_0 \in K^0$. Therefore we can find $\alpha \in (0, r(T))$ such that $Tz_0 \geq \alpha v_0$, i.e.,

$$(r(T) - \alpha)v_0 + (\xi_1 u + \eta_1 v) \geq 0, \tag{1.2}$$

where

$$\xi_1 = \xi_0\sigma + \eta_0\tau,\ \eta_1 = \eta_0\sigma - \xi_0\tau.$$

Clearly

$$\xi_1^2 + \eta_1^2 = (\sigma^2 + \tau^2)(\xi_0^2 + \eta_0^2) = M|\lambda|^2.$$

By (1.2), we find that $(\xi_1, \eta_1)/(r(T) - \alpha) \in \Sigma$ and hence

$$\xi_1^2 + \eta_1^2 \le M(r(T) - \alpha)^2,$$

that is,

$$|\lambda|^2 \le (r(T) - \alpha)^2,$$

and hence $|\lambda| < r(T)$. The proof of Step 4 and hence the theorem is now complete. □

Suppose now L is the elliptic operator and Ω the bounded domain as given in Theorem A.4, namely

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$$

has $C^\alpha(\overline{\Omega})$ coefficients and is strictly uniformly elliptic in the bounded domain Ω which has $C^{2,\alpha}$ boundary. Choose $\xi > 0$ large enough so that $c - \xi < 0$ in Ω, and denote $L_\xi u = Lu - \xi u$. Let K be the positive cone in $X := C_0^{1,\alpha}(\overline{\Omega})$ consisting of nonnegative functions. For any $v \in X$, Theorem A.1 guarantees that the problem

$$-L_\xi u = v \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

has a unique solution u satisfying

$$\|u\|_{2,\alpha} \le C\|v\|_\alpha \le C_1\|v\|_{1,\alpha}$$

for some constant $C_1 > 0$ independent of u and v. It follows that $T : X \to X$ defined by $Tv = u$ is a compact linear operator. Moreover, by the weak maximum principle Theorem A.34, $Tv \ge 0$ if $v \in K$. The strong maximum principle Theorem A.36 then implies that $u = Tv > 0$ in Ω if $v \in K \setminus \{0\}$, and the Hopf boundary lemma (Lemma A.35) gives further $D_\nu u < 0$ on $\partial\Omega$. This implies that $Tv \in K^0$. Therefore T is strongly positive. It now follows from Theorem 1.2 that $r(T) > 0$ is a simple eigenvalue of T with an eigenfunction $v \in K^0$: $Tv = r(T)v$. Thus $u = Tv$ satisfies

$$-Lu + \xi u = r(T)^{-1}u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

i.e.,

$$Lu + \lambda_1 u = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

with $\lambda_1 = r(T)^{-1} - \xi$.

Generally, it is easily checked that μ is an eigenvalue of T if and only if $\lambda = \mu^{-1} - \xi$ is an eigenvalue of

$$Lu + \lambda u = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Theorem 1.2 now implies the following result.

Theorem 1.3 *Under the conditions of Theorem A.4 for L and Ω, the eigenvalue problem (1.3) has a simple eigenvalue $\lambda_1 \in R^1$ which corresponds to a positive eigenfunction; none of the other eigenvalues corresponds to a positive eigenfunction.*

If the boundary operator is of Neumann or Robin type,

$$Bu = D_\nu u + \sigma(x)u, \ \sigma \geq 0, \ \sigma \in C^{1,\alpha}(\partial\Omega),$$

then we let $X = C^{1,\alpha}(\overline{\Omega})$ and let K be the cone of nonnegative functions in this space. We define the operator T analogously as in the Dirichlet case and again find that it is compact on X and maps K to itself, due to the weak maximum principle. Suppose now $v \in K \setminus \{0\}$. Then by the strong maximum principle, $u = Tv > 0$ in Ω. Moreover, by the Hopf boundary lemma, if $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then $D_\nu u(x_0) < 0$ and hence $Bu(x_0) < 0$, contradicting the boundary condition. Therefore $u > 0$ on $\partial\Omega$. Therefore $Tv > 0$ on $\overline{\Omega}$, which implies that $Tv \in K^0$, i.e., T is strongly positive. Therefore we can apply Theorem 1.2 to conclude that Theorem 1.3 holds also for the Neumann and Robin boundary conditions. The eigenvalue λ_1 in Theorem 1.3 is usually called the *principle eigenvalue.*

Theorem 1.4 *If $\lambda \neq \lambda_1$ is an eigenvalue of (1.3) but the boundary condition is either Dirichlet, or Neumann, or Robin type, then $Re(\lambda) \geq \lambda_1$.*

Proof. Suppose $w > 0$ is an eigenvector corresponding to λ_1 and u is an eigenvector corresponding to λ. Set $v := u/w$. Then

$$-\lambda v = w^{-1}L(vw) = Lv - cv + 2w^{-1}a^{ij}D_j w D_i v - \lambda_1 v.$$

Writing

$$Kv := a^{ij}D_{ij}v + \tilde{b}^i D_i v, \ \tilde{b}^i := b^i + 2w^{-1}a^{ij}D_j w,$$

we obtain

$$Kv + (\lambda - \lambda_1)v = 0.$$

Take complex conjugates to yield

$$K\overline{v} + (\overline{\lambda} - \lambda_1)\overline{v} = 0.$$

Next we compute

$$K(|v|^2) = K(v\overline{v}) = \overline{v}Kv + vK\overline{v} + 2a^{ij}D_ivD_j\overline{v} \geq \overline{v}Kv + vK\overline{v},$$

since

$$a^{ij}\xi_i\overline{\xi}_j = a^{ij}(Re(\xi_i)Re(\xi_j) + Im(\xi_i)Im(\xi_j)) \geq 0$$

for any complex vector $\xi \in C^N$. We now easily obtain

$$K(|v|^2) \geq 2(Re(\lambda) - \lambda_1)|v|^2 \text{ in } \Omega.$$

Suppose now the boundary operator B is either Neumann or Robin type. Then $w > 0$ over $\overline{\Omega}$ and a direct computation shows $D_\nu v = 0$ and $D_\nu |v|^2 = 0$. If $Re(\lambda) \leq \lambda_1$, then $\phi := |v|^2 \geq 0$ satisfies

$$K\phi \geq 0 \text{ in } \Omega,\ D_\nu \phi = 0 \text{ on } \partial\Omega.$$

We now apply the strong maximum principle and Hopf boundary lemma and conclude that $\phi \equiv$ constant, that is $u = cw$ and hence $\lambda = \lambda_1$, a contradiction. Therefore we must have $Re(\lambda) > \lambda_1$.

To prove the Dirichlet case, we replace w by $w_\epsilon := w^{1-\epsilon}$, $0 < \epsilon < 1$, in the above discussion and obtain

$$K(|v|^2) \geq -2(Re(\lambda) + \frac{Lw_\epsilon}{w_\epsilon})|v|^2 \text{ in } \Omega.$$

Since

$$Lw_\epsilon = (1-\epsilon)w^{-\epsilon}Lw - \epsilon(1-\epsilon)w^{-1-\epsilon}a^{ij}D_iwD_jw + \epsilon c w^{1-\epsilon}$$

$$\leq (1-\epsilon)w^{-\epsilon}Lw + \epsilon c w^{1-\epsilon} \leq (\epsilon C - (1-\epsilon)\lambda_1)w_\epsilon,$$

where $C = \max_{\overline{\Omega}} c$, we deduce

$$K(|v|^2) \geq 2((1-\epsilon)\lambda_1 - \epsilon C - Re(\lambda))|v|^2 \text{ in } \Omega.$$