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CONTENTS

ABSTRACT

A theory of almost periodic measures on locally compact abelian groups is developed which includes as special cases the original concept of Bohr [11], as well as most of the subsequent generalizations at the hands of Wiener [61], Stepanoff [52], Bohr and Besicovitch [9] and Eberlein [24]. The theory is developed in the context of a (topological) translation module over G , which is, among other things, a locally convex space of measures on G . The almost periodicity of a measure is defined in terms of the Bochner compactness criterion for the set of translates. The classical theory of Bohr centering about the mean and the Fourier expansions is developed and the connection between almost periodicity and the general Fourier transform [2] introduced earlier by us is explored. The theory is applied to the explanation and further development of work centering about mixed normed spaces, beginning with Wiener and continued more recently by Bertrandias et al. [7,8], Busby and Smith [16] and Stewart [56].

NOTE

A short supplement has been added at the end of this work, just before the bibliography. The first four items are clarifications of a few parts of the main text, in order to make the present work more accessible to non-specialists. The last item contains references to the literature which were not available to the authors until the main text was completed.

These items are preceded by an increasing number of asterisks, and any passage in the main text covered by a given item is marked on the right-hand margin by the same number of asterisks as the pertinent item. Thus *** on the margin marks a passage in the main text for which additional information is contained in Item *** in the supplement.

A supplementary bibliography is also attached after the main bibliography. Each item is preceeded by S and a number. Reference to these items, which occur in the supplement, are labeled accordingly. Thus [S-3] is Item 3 in the supplementary bibliography.

Partial financial support for various stages of this work has been derived from the U.S. National Science Foundation, the University of Nebraska Research Council and the Deutsche Forschungsgemeinschaft. A preliminary version of portions of the material was presented at the AMS regional meeting in Chicago on March, 1971, as well as at various symposia and colloquia, listed in the introduction.

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1. Introduction. The present work was originally conceived as a sequel to [2], where a general theory of the Fourier transform of (often unbounded) measures was developed, a kind of tempered distribution theory for measures on locally compact abelian groups, which includes most of the classical notions of the Fourier transform. Our main purpose then was to elaborate the conceptual framework and the theoretical machinery needed to formulate and study the almost periodic behavior of the new general Fourier transform, as suggested by the theorem of Eberlein [25] stating that the Fourier-Stieltjes transform of a finite (bounded) measure is weakly almost periodic. A general theory of almost periodic measures was developed which closely paralleled and included the classical theory of Bohr of almost periodic continuous functions, and the subsequent generalizations by Bochner and von Neumann to topological groups.

As the work progressed, other notions of almost periodicity came to our attention, which had been in the literature for a long time, and which we had not originally envisaged in our work. These notions were primarily the pseudo-periodic functions of Wiener [61] and the more general treatment of Stepanoff [52] and further generalizations at the hands of Bohr himself and Besicovitch (See Chapter II of Besicovitch's book [9]). The extensions of Wiener, Stepanoff, Bohr and Besicovitch, and possibly others to discontinuous almost periodic functions, remained to this day essentially confined to $\mathbb{R}$. New impetus was provided by the recent work of Holland [35], Bertrandias et al. [7], Stewart [56] and Busby and Smith [16] on mixed norm spaces, raising the possibility of extending the theory to general locally compact abelian groups. It soon became clear that our methods, originally conceived in a narrower context to deal primarily with the Fourier transform, after a suitable recasting, could be developed into a theory which encompassed most of the classical phenomena of almost periodicity. The result has been the general concept of almost periodicity of measures discussed here.

The principal vehicle for our theory is a translation module over a locally compact abelian group G, which is a translation invariant vector space $\mathfrak{M}$ of measures (finite or not) on G, which is a locally convex topological vector space (l.c.t.v.s.) with respect to some suitable topology. An almost periodic measure in $\mathfrak{M}$ is one for which the set of translates has compact closure. Different classical and new notions of almost periodicity are obtained by varying $\mathfrak{M}$ or its topology. We have resisted the temptation of adopting the next obvious generalization by taking $\mathfrak{M}$ to be an abstract l.c.t.v.s., not necessarily of measures, on which G acts. Such a notion has appeared to us to be too general at this stage, in that it is not clear to us how some of the more powerful concrete analytic results in the theory of Bohr can be formulated and proved in such a context, whereas most go over with remarkable ease and simplicity for our notion of almost periodic measure. To be sure, some of the nice results admit striking generalization as in the work on almost periodic vectors in Banach spaces at the hands of Eberlein [24] and Jacobs [39]. Similar fears of the limitations on the scope and power of the resulting theory imposed by excessive generality led us to avoid the Schwartz-Bruhat distributions, instead of the more concrete approach by measures, although for R^n Schwartz [50] does have a notion of almost periodicity for distributions and has obtained some striking results which we have used as inspiration. In both the vector and the distribution approach the major classical results admitting simple and striking generalizations appear to us to be the exception rather than the rule.

Thus, our theory is more general than the classical notions of almost periodicity and includes most of them. The connection with almost periodic distributions in the case of $G = \mathbb{R}^n$ is not completely clear to us, because of the different topologies used, but the present work goes much further. It is less general than the almost periodic vectors in that our vectors are measures, but more general in that our G-modules are l.c.t.v.s., not just Banach spaces.

In the next section we lay down the basic terminology and notation that will be used throughout the rest of the work and introduce various concrete spaces of measures (and functions) with their natural topologies and study their basic properties. These spaces will furnish the various examples of our general concept of almost periodicity. The various mixed norm spaces $L^{p,q}(G)$ are recalled [7,16,35,56] as well as the related spaces $\mathfrak{M}^q(G)$ of measures. Parts of their basic theory are developed to meet the requirements of the present work. In the same section the theory of the space $\mathfrak{M}^\infty(G)$ of translation bounded measures, begun in [2], is carried further.

In Section 3 the concept of translation module over G (G-module) is introduced and studied. This is a translation invariant vector space of measures on G which is a l.c.t.v.s. with respect to a topology which is stronger (finer) than the vague topology. The notion of translation continuity and its basic properties, as well as the convolution theorem discussed in that section are basic in the rest of the work.

In Section 4 we introduce **amenable measures** (essentially measures for which the mean exists in the sense of Bohr) are introduced. Much of the section is devoted to finding methods of computing the mean of amenable measures by means of various generalizations of the classical Bohr formula for the mean of an almost periodic function on $\mathbb{R}$.

In Section 5 we introduce the concept of almost periodicity of a measure in a G-module. The basic theory of these measures is developed and the applications to the various classical and not so classical examples examined. By identifying functions with absolutely continuous measures we encompass the various classical notions of almost periodicity, from Bohr's original notion to the Wiener-Stepanoff concept. It turns out that $\mathcal{M}^{\infty}(G)$ is the largest G-module of measures on G and many abstract questions involving almost periodicity can be reduced to this relatively concrete case.

In Section 6 we exploit the stability of certain G-modules, $\mathcal{M}^{\infty}(G)$ in particular, with respect to pointwise multiplication of measures by uniformly continuous functions, to lay the groundwork for the material in Sections 7, 8 and 9 on the Fourier-Bohr series and the Eberlein-type decomposition of a weakly almost periodic measure as the sum of a strongly almost periodic measure and of a null weakly almost periodic measure. In this circle of ideas a mapping is defined which enables us to identify an almost periodic measure with a measure on the Bohr compactification G_b of G , in analogy with the classical identification of an almost periodic function on G with a continuous function on G_b .

Sections 10 and 11 are devoted to applications of our theory to the study of our general Fourier transform of measures [2]. As a by-product we obtain improvements and extensions to general locally compact groups of results of Wiener [61] and Holland [36,37] for $G = \mathbb{R}$.

Our last section is devoted to the discussion of the behavior of our general concept of almost periodicity of measures with respect to the passage to subgroups (restriction) and quotient groups (quotient measures). Our results are only partially complete, but aid in the construction of non-standard examples of the various phenomena encountered here.

A work of the scope of the present one, developed over many years, owes its existence to many favorable influences, some of which we are now pleased to acknowledge. Some of this material was developed while the second named author was a visiting professor at the University of Rennes (through the kind hospitality of A. Brunel, M. Keane and M. Métivier), at the University of Munich (through the kind hospitality of G. Hämmerlin and E. Wienholtz) and at the Indian Statistical Institute in Calcutta (under the kind hospitality of G. Kallianpur).

We would like to record a special word of thanks to Ms. Kathy Swedell for exceptional professionalism and care in the typing of the entire manuscript.

In addition to the presentation at the AMS regional meeting of March, 1971, cited on the title page, preliminary portions of the present work were presented at the Maryland Conference on Harmonic Analysis, 1972 [3], the KGB Seminar at Rennes, 1972 [29], the Vienna Winterschule, 1979 (Internationale Arbeitstagung über topologische Gruppen und Gruppenalgebren, Institut für Mathematik der Universität Wien, Tagungsbericht), and the Hille Laguna Beach Conference, January, (1980).

2. Preliminaries. We begin this section by setting down the basic terminology and notations of standard abstract harmonic analysis as they are going to be employed in the rest of the work. We then proceed to draw up a catalogue of various spaces of functions and of measures. These spaces will serve later on to illustrate concretely various phenomena appearing in our more abstract theory of translation stable spaces of measures introduced in the next section. As the discussion proceeds we introduce new notations and conventions appropriate to our discussion here and explain the point of view adopted here regarding infinite (unbounded) measures and their convolution.

Much of the basic material is drawn, with at most trivial modifications, from standard treatises (e.g., [32]). Other parts, such as the spaces $L^{p,q}(G)$ come from the more recent literature ([16,7,56]). At the end we establish new results needed in the sequel concerning spaces of translation bounded measures, introduced by us in [2].

Throughout the entire work G will stand for a locally compact abelian group denoted multiplicatively with identity e and ω for a Haar measure, for the most part regarded as predetermined and fixed. In integrals the infinitesimal increment of ω is denoted as usual by dx. By the elementary operations on a function f we shall mean the usual operations of complex conjugation, reflection $f \to f'$, defined by

$$(2.1) \qquad f'(x) = f(x^{-1})$$

and the usual involution $f \to f^* = (f')^- = (\overline{f})'$.

Let us begin by drawing up our catalogue of spaces of functions and measures. As we proceed we develop the basic notions, conventions and notations pertaining to these spaces and to their individual elements. For easy future reference we label each space as a numbered example. Unless otherwise specifically indicated, all scalars are complex and all functions are complex valued.

EXAMPLE 2.1. For $1 \leqq p \leqq +\infty$, the l.c.t.v.s. $L^p_{loc}(G)$ of locally L^p-functions with the topology of local L^p-convergence (L^p-convergence on compact sets).

EXAMPLE 2.2. The subspace $C(G)$ of $L^\infty_{loc}(G)$ consisting of continuous functions with the relative local L^∞-convergence topology.

EXAMPLE 2.3. The subspace $C_B(G)$ of $C(G)$ consisting of bounded functions, but with the Banach space sup norm topology.

EXAMPLE 2.4. The closed subspace $C_U(G)$ of the Banach space $C_B(G)$ under the relative sup norm topology, consisting of uniformly continuous functions.

EXAMPLE 2.5. The closed subspace $C_o(G)$ of $C_U(G)$ consisting of all continuous functions vanishing at ∞.

EXAMPLE 2.6. The space $L^p_{comp}(G)$ of all L^p-functions with compact support with the inductive limit topology defined on it by the (Banach) subspaces $L^p(A,G)$ of L^p-functions supported by a fixed compact set $A \subset G$ under the L^p-norm topology. For $p>1$, $L^p_{comp}(G)$ is the dual of the l.c.t.v.s. $L^{p'}_{loc}(G)$ of Example 2.1, where p' is the conjugate index of p. For $1 \leqq p < +\infty$, the dual of the l.c.t.v.s. $L^p_{comp}(G)$ is $L^{p'}_{loc}(G)$. The duality between local L^p-spaces is the usual one:

$$\langle f,g \rangle = \int_G f(x)g(x)dx \quad . \tag{2.2}$$

EXAMPLE 2.7. The sub-space $K(G)$ of $L^\infty_{comp}(G)$ consisting of continuous functions with the relative inductive limit topology.

EXAMPLE 2.8. The vector subspace $K_2(G)$ of $K(G)$ spanned by functions of the form $f*g$, $f, g \in K(G)$ with the relative inductive limit topology.

EXAMPLE 2.9. The Banach mixed norm space $L^{p,q}(G)$ for $1 \leqq p,q \leqq +\infty$, with any of the (equivalent) mixed norms. These spaces, already implicit in the work of Wiener [61], Stepanov [52] and Besicovitch [9] for $G = R$ and $q = +\infty$, were introduced formally by Holland [35], again for $G = R$. The notion was later on generalized more or less simultaneously by Bertrandias, Datry and Dupuis [7] (G abelian), Stewart [56] (G compactly generated) and Busby and Smith [16]. In the discussion of these spaces, which we now begin, we follow to varying extents the presentation, terminology and notations of Busby and Smith from whom we first learned about these spaces for general G . For details omitted here we refer the reader to [16].

We begin with a uniform partition $\pi = \{E\}$ of G . This, among other things [16, Definition 3.1], is a covering of G by disjoint relatively compact sets E with non empty interior. We consider π as a discrete space and on it the space $\ell^q(\pi)$. For $f \in L^p_{loc}(G)$, we write ${}_E\|f\|_p$ for the L^p-norm of f on E , and denote by $\|f\|^\pi_{p,q}$ the $\ell^q(\pi)$-"norm" of $\{{}_E\|f\|_p\}_{E\in\pi}$ as a function on π . The space $L^{p,q}(G)$ consists exactly of those $f \in L^p_{loc}(G)$, with *

$$(2.3) \qquad \|f\|^\pi_{p,q} < +\infty \; .$$

On $L^{p,q}(G)$, $\|f\|^\pi_{p,q}$ is a norm which turns it into a Banach space. If we use a different uniform partition, the norm will change, but the space and the Banach space topology do not and in this sense the space $L^{p,q}(G)$ is independent of the uniform partition π . For easy reference we state now a characterization of the spaces $L^{p,q}(G)$, due to Busby and Smith [16, Proposition 3.9], for $1 \leqq p < +\infty$.

THEOREM 2.1. Let $1 \leqq q \leqq +\infty$ and $1 \leqq p < +\infty$ and θ an arbitrary, but fixed function in $K(G)$ with $0 \neq \theta \geqq 0$. Then, for any $f \in L^p_{loc}(G)$, $f \in L^{p,q}(G)$ if and only if $[\theta * |f|^p]^{1/p} \in L_q(G)$ and

(2.4) $$\|f\|_{p,q} = \| |\theta * |f|^p|^{1/p}\|_q$$

defines a norm on $L^{p,q}(G)$ equivalent to the π-norms.

We are now going to give an alternative but similar characterization which we believe is new, and which has the advantage of being valid also for $p = +\infty$. The proof, however, is similar to that of the above Theorem 2.1 given in [16]. Let us denote by $\delta_x * f$ the translate

(2.5) $$\delta_x * f(y) = f(yx^{-1})$$

of the function f by $x \in G$. Throughout this work we denote by f_A the restriction of the function f to the set A. Thus

(2.6) $$f_A = 1_A f \quad ,$$

where 1_A is the indicator (characteristic) function of the set A. In the next theorem it will be convenient to have the following notation. For a fixed $g \in K(G)$, with $0 \neq g \geqq 0$, and $1 \leqq p \leqq +\infty$, given a measurable function f on G, it is clear that the quantity

(2.7) $$h(x) = \|(\delta_x * g)f\|_p$$

is well defined (finite) for every $x \in G$ if and only if $f \in L^p_{loc}(G)$. In fact, in this case, h is a continuous function of x, even for $p = +\infty$.

THEOREM 2.2. Let $1 \leqq p,q \leqq +\infty$ and g an arbitrary but fixed ***** function in $K(G)$ with $0 \neq g \geqq 0$. Then, for any measurable function f on G, $f \in L^{p,q}(G)$ if and only if the function h is well defined by (2.7) and belongs to $L^q(G)$. Further, the quantity

$$(2.8) \qquad {}_g\|f\|_{p,q} = \|h\|_q$$

defines a norm ${}_g\| \ \|_{p,q}$ on $L^{p,q}(G)$ which is equivalent to all heretofore given norms on $L^{p,q}(G)$.

Proof. Clearly, what has to be shown is that, for any fixed uniform partition π of G, there are positive numbers α and β such that

$$(2.9) \qquad \alpha\|f\|^{\pi}_{p,q} \leqq {}_g\|f\|_{p,q} \leqq \beta\|f\|^{\pi}_{p,q}$$

for every measurable f, whether the quantities in (2.9) are finite or not.

We prove the last inequality in (2.9). The following argument as given is strictly speaking valid only for $q < +\infty$, but an obvious slight modification which we omit makes it valid for the exceptional case $q = +\infty$. Clearly, we may assume that $\|f\|^{\pi}_{p,q} < +\infty$. In this case f is locally integrable and h is well defined and continuous. Now for $x \in E \in \pi$,

$$(2.10) \qquad h(x) = \|(\delta_x * g)f\|_p \leqq \|g\|_\infty \|f_{xK^{-1}}\|_p \leqq \|g\|_\infty \|f_{EK^{-1}}\|_p \quad ,$$