BEIJING LECTURES IN HARMONIC ANALYSIS

EDITED BY

E. M. STEIN

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PREFACE

In September 1984 a summer school in analysis was held at Peking University. The subjects dealt with were topics of current interest in the closely interrelated areas of Fourier analysis, pseudo-differential and singular integral operators, partial differential equations, real-variable theory, and several complex variables. Entitled the "Summer Symposium of Analysis in China," the conference was organized around seven series of expository lectures whose purpose was to give both an introduction of the basic material as well as a description of the most recent results in these areas. Our objective was to facilitate further scientific exchanges between the mathematicians of our two countries and to bring the students of the summer school to the level of current research in those important fields.

On behalf of all the visiting lecturers I would like to acknowledge our great appreciation to the organizing committee of the conference: Professors M. T. Cheng and D. G. Deng of Peking University, S. Kung of the University of Science and Technology of China, S. L. Wang of Hangzhou University, and R. Long of the Institute of Mathematics of the Academia Sinica. Their efforts helped to make this a most fruitful and enjoyable meeting.

E. M. STEIN

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Beijing Lectures in

Harmonic Analysis

NON-LINEAR HARMONIC ANALYSIS, OPERATOR THEORY AND P.D.E.

R. R. Coifman and Yves Meyer

Our purpose is to describe a certain number of results involving the study of non-linear analytic dependence of some functionals arising naturally in P.D.E. or operator theory.

To be more specific we will consider functionals i.e., functions defined on a Banach space of functions (usually on R^n) with values in another Banach space of functions or operators.

Such a functional $F:B_1 \to B_2$ is said to be real analytic around 0 in B_1 if we can expand it in a power series around 0 i.e.

$$F(f) = \sum_{k=0}^{\infty} \Lambda_k(f)$$

where $\Lambda_k(f)$ is a "homogeneous polynomial" of degree k in f. This means that there is a k multilinear function

$$\Lambda_k(f_1 \cdots f_k) : B_1 \times B_1 \cdots \times B_1 \to B_2$$

(linear in each argument) such that $\Lambda_k(f) = \Lambda_k(f, f, \cdots f)$ and

(1) $$\|\Lambda_k(f_1 \cdots f_k)\|_{B_2} \leq C^k \prod_{j=1}^{k} \|f_j\|_{B_1}$$

for some constant C. (This last estimate guarantees the convergence of the series in the ball $\|f\|_{B_1} < \frac{1}{C}$.)

Certain facts can be easily verified. In particular if F is analytic it can be extended to a ball in B_1^C (the complexification of B_1) and the extension is holomorphic from B_1^C to B_2^C i.e., $F(f+zg)$ is a holomorphic (vector valued) function of $z \in \mathbb{C}$, $|z| < 1$, $\forall f, g$ sufficiently small. The converse is also true. Any such holomorphic function can be expanded in a power series, (where Λ_k is $\frac{1}{k!} \times$ the k^{th} Frechet differential at 0).

We will concentrate our attention on very concrete functionals arising in connection with differential equations or complex analysis, and would like to prove that they depend analytically on certain functional parameters.

As you know there are two ways to proceed.

1. Expand in a power series and show that one has estimates (1).

2. Extend the functional to the complexification as "formally holomorphic" and prove some boundedness estimates.

Let L denote a differential operator like

$$a(x) \frac{d}{dx} \qquad x \in \mathbf{R},$$

$$a(z) \frac{\partial}{\partial z} \qquad z \in \mathbf{C}$$

$$\sum \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} = \operatorname{div} A(x) \operatorname{grad}, \quad A = (a_{ij}) \qquad x \in \mathbf{R}^n$$

$$\sum a_{ij}(x) \frac{\partial}{\partial x_i \partial x_j} \qquad x \in \mathbf{R}^n$$

the coefficients $a(x)$ (or $a_{ij}(x)$) will be assumed to belong to some Banach space B_1 of functions (for example L^∞). It is natural to ask when such objects as:

$$L^{-1}, \ \sqrt{L}, \ \operatorname{sgn} L, \ e^{-tL}, \ e^{-t\sqrt{L}}$$

or more generally, $\phi(L)$ (where $\phi: \mathbf{C} \to \mathbf{C}$), can be defined as a bounded operator (say on L^2 or some Soboleff space), and a functional calculus developed i.e., $\phi_1(L)\phi_2(L) = \phi_1\phi_2(L)$.

Many questions arise:

a) Does $F(a) = \phi(L)$ viewed as an operator valued function of a depend analytically on a ?

This is equivalent to asking whether we can consider complex valued coefficients in L and still have estimates on $\phi(L)$.

b) What is the largest domain of coefficients a for which we have estimates for $\phi(L)$? This question is the same as asking what is the *largest* B_1 for which (1) holds, and what is the domain of holomorphy of $F(a)$ in this space.

The answer to question a) will require first that we understand methods for expanding functionals in a power series, and second, that the nature of the multilinear operators Λ_k be sufficiently well understood to provide estimates (1). As for question b) we will see that the largest spaces possible for the coefficients involve rough coefficients and leads us to work with coefficients in L^∞, B.M.O. and other "exotic spaces."

We now start with a fundamental example related to the Cauchy integral. We let

$$L_a = \frac{1}{1+a}\frac{1d}{idx} \quad \text{with } \|a\|_\infty < 1 \quad a(x), \text{ real valued.}$$

If we define $h(x) = x + A(x)$, $A'(x) = a$. We then have

$$L_a f = \left(\frac{1d}{idx} f \circ h^{-1}\right) \circ h = \frac{1}{i} U_h \frac{d}{dx} U_h^{-1} f$$

where

$$U_h f = f \circ h .$$

Of course, in this case, if we use the Fourier transform we can define

$$\phi\left(\frac{1}{i}\frac{d}{dx}\right) f = \int_{-\infty}^{\infty} e^{ix\xi}\phi(\xi)\hat{f}(\xi)\, d\xi .$$

This gives, for example

$$\operatorname{sgn}\left(\frac{1}{i}\frac{d}{dx}\right)f = \int e^{ix\xi}\operatorname{sgn}\xi\,\hat{f}(\xi)\,d\xi = \frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt = H(f)\,.$$

Thus we can define

$$\pi\operatorname{sgn}(L_a)f = \pi U_h \operatorname{sgn}\frac{1}{i}\frac{d}{dx}\,U_h^{-1}f = \text{p.v.}\int_{-\infty}^{\infty}\frac{f(t)\,(1+a(t))}{x-t+A(x)-A(t)}\,dt$$

(where we used the observation that $\phi(ULU^{-1}) = U\phi(L)U^{-1}$).

We view

$$F(a) = \operatorname{sgn} L_a \quad \text{as an operator on } L^2(R)$$

and wish to know whether it is analytic on L^∞ or if we can replace a by complex a and still have a bounded operator.

If we do this, writing $a = \alpha + i\beta$ $\|a\|_\infty < 1$, we find

$$F(a)f = \int\frac{f(t)\ \ (1+i\alpha+i\beta)}{x-t+A(x)+iB(x)-A(t)-iB(t)}\,dt$$

$$= \int\frac{f(t)[(1+a)/(1+\alpha)](1+\alpha)}{x+A(x)-t-A(t)+i(B(x)-B(t))}\,dt$$

$$= U_h C U_h^{-1} f_1$$

where

$$h = x+A(x),\ Cf = \int_{-\infty}^{\infty}\frac{f(t)\ \ (1+B_1'(t))}{x-t+iB_1(x)-iB_1(t)}\,dt$$

$$f_1(t) = f(t)\,\frac{1+a}{1+\alpha}\,\frac{1}{B_1'(t)} \qquad B_1 = B\circ h^{-1}\,.$$

Since U_h is bounded on L^2 it would suffice to prove that C is bounded on L^2 for all B such that B' is small.

We could also try to prove this by expanding

$$-i\pi \operatorname{sgn}(L_a)f = \int \frac{f(t)\ (1+a(t))}{(x-t)+A(x)-A(t)}\,dt = \sum (-1)^k \int_{-\infty}^{\infty} \left(\frac{A(x)-A(t)}{x-t}\right)^k \frac{f(1+a)}{x-t}\,dt .$$

Observe that the operators are of the form

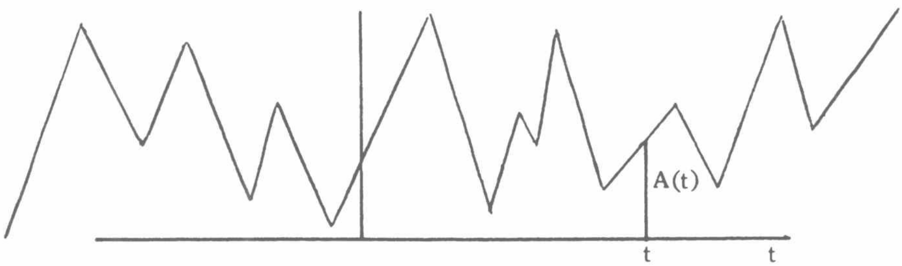
$$T(f) = \int \Psi\left(\frac{A(x)-A(t)}{x-t}\right)\frac{f(t)}{x-t}\,dt = \int k(x,t)\,f(t)\,dt .$$

We will prove Theorem I: Let $\Psi \in C^\infty(C)$ and $A(x)$ such that

$$\left|\frac{A(x)-A(t)}{x-t}\right| \le M \quad\text{and}\quad T(f) = \text{p.v.}\int \Psi\left(\frac{A(x)-A(y)}{x-y}\right)\frac{f(y)}{x-y}\,dy .$$

Then the operator T is bounded on $L^2(R)$ (and L^p $1 < p < \infty$). This result will then be extended to R^n and other settings.

We now return to the interpretation of C as the Cauchy integral for the curve $z(t) = t + iA(t)$ where A is Lipschitz



as we can see its boundedness in L^2 is equivalent to the analytic dependence of $C(a)f$ on the curve a. This now is related to the lectures by C. Kenig (to which we shall return later).

Let us consider a more general version of the Cauchy integral.

Let Γ be a rectifiable curve through 0, s be the arc length parameter

$$z'(s) = e^{i\alpha(s)} \text{ i.e., } z(s) = \int_0^s e^{i\alpha(t)}dt .$$

The Cauchy integral on Γ is given as:

$$C_\Gamma(f) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)z'(t)}{z(s)-z(t)}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\frac{z(s)-z(t)}{s-t}} \frac{f(t)z'(t)}{s-t}\,dt$$

$$= \int_{-\infty}^{\infty} \Psi\left(\frac{z(s)-z(t)}{s-t}\right) \frac{f_1(t)}{s-t}\,dt$$

if we assume

$$* \qquad 0 < \delta < \left|\frac{z(s)-z(t)}{s-t}\right| \leq 1$$

we can take $\psi \in C_0^\infty(C)\psi(z) = \frac{1}{z}$ on $\delta < |z| \leq 1$ and obtain the boundedness on L^2 of C_Γ (from Theorem I).

Condition * is the so-called chord arc condition and * for δ small is equivalent to $\alpha \in$ BMO with $\|\alpha\|_{BMO}$ small (see [3]). If we think of C as an operator valued functional of α, we will see that B.M.O. is the space of analyticity or holomorphy of C_α.

§2.

All the operators which we encountered previously had the form

$$T(f) = \text{p.v.} \int K(x,y) f(y)\, dy$$

where $|K(x,y)| < \frac{C}{|x-y|}$ $|\partial_x K| + |\partial_y K| \leq \frac{C}{|x-y|^2}$, moreover they were also antisymmetric i.e.,

$$K(x,y) = -K(y,x) .$$

(For example, $K(x,y) = \phi\left(\frac{A(x)-A(y)}{x-y}\right)\frac{1}{x-y}$ $\phi \in C^4$, $A' \in L^\infty$.) Recently G. David and J.-L. Journé found a necessary and sufficient condition for such operators to be bounded on L^2 (or L^p). This condition is simply that $T(1)$ must be of bounded mean oscillation.

We now would like to state certain facts concerning B.M.O. and prove their theorem.

Recall that $b \in BMO(R)$ if

$$\|b\|_* = \left(\sup_I \frac{1}{|I|} \int_I |b - m_I(b)|^2 dx\right)^{1/2} < \infty, \text{ where } m_I(b) = \frac{1}{|I|} \int_I b(x)\, dx$$

and I is an interval (or a cube in R^n), and that this norm is equivalent to the following "Carleson" norm

$$\sup_I \left(\frac{1}{|I|} \int_I \int_0^{|I|} |\psi_t * b|^2 \frac{dx\, dt}{t}\right)^{1/2}$$

where $\psi_t = \frac{1}{t}\psi\left(\frac{x}{t}\right)$ $\psi \in C_0^\infty$, $\int \psi dx = 0$ $(\psi \not\equiv 0)$ (see [5]).

A basic reason for the frequent occurrence of functions in B.M.O. is the following simple fact.

PROPOSITION. *If* T *is as above and* T *is bounded on* L^2 *then* T *maps* L^∞ *into* B.M.O.

Proof. Let $b \in L^\infty$ and let I be given. Consider $\overline{I} = 2I$ and write $\widetilde{b} = Tb = T(b\chi_{\overline{I}}) + T(b(1-\chi_{\overline{I}})) = \widetilde{b}_1 + \widetilde{b}_2$.

Clearly

$$\left(\frac{1}{|I|}\int_I |\widetilde{b}-m_I(\widetilde{b})|^2\right)^{1/2} \le \left(\frac{1}{|I|}\int_I |\widetilde{b}_1-m_I(\widetilde{b}_1)|^2\right)^{1/2} + \left(\frac{1}{|I|}\int_I |\widetilde{b}_2-m_I(\widetilde{b}_2)|^2\right)^{1/2}.$$

The first term is dominated by

$$2\left(\frac{1}{|I|}\int_I |\widetilde{b}_1|^2 dx\right)^{1/2} \le C\left(\frac{1}{|I|}\int_I |b_1|^2\right)^{1/2} \le C\|b\|_\infty .$$

For the second we observe that

$$T(b_2)(x) - T(b_2)(u) = \left|\int [K(x,y)-K(u,y)]b_2(y)dy\right|$$

$$\le \int_{|x-y|>|I|} \frac{|I|}{|x-y|^2}\, dy\|b\|_\infty \le C\|b\|_\infty .$$

Integrating in y we get

$$|T(b_2)(x) - m_I(T(b_2))| < C\|b\|_\infty$$

which shows that second term is bounded by $C\|b\|_\infty$. We have thus shown the necessity of the condition $T(1) \in BMO$. Before stating the theorem precisely we would like to reformulate it somewhat.

Let $\phi \in C_0^\infty(R^1)$ with $\int \phi dx = 1$ and $\psi = \phi(x)$. Let $\phi_t^x(u) = \frac{1}{t}\phi\left(\frac{u-x}{t}\right)$ and similarly for $\psi_t^x(u)$. We claim that under the preceding assumptions on T we have

$$|<T\psi_t^x, \phi_t^y>| \le CP_t(x-y) = C\frac{1}{t}\frac{1}{1+\left(\frac{x-y}{t}\right)^2} .$$

In fact, assume for simplicity, that ϕ is supported in $(-1,1)$. Since $\int \psi du = 0$, if we assume $|x-y| > 3t$

$$|<T\psi_t^x, \phi_t^y>| = \left|\int \phi_t^y(z)\left\{\int [K(z,u) - K(z,x)]\psi_t^x(u)\, du\right\} dz\right|$$

$$\le \int |\phi_t^y(z)| \frac{t}{|y-x|^2} |\psi_t^x(u)|\, du\, dz \le C\frac{t}{|y-x|^2}$$

(where we used the fact that $|y-z| < t$, $|x-u| < t$, $|x-y| > 3t$ and the hypothesis $|\partial_y K(x,y)| \le |x-y|^{-2}$).

If $|x-y| < 3t$ we use the antisymmetry of $k(x,y)$ to write

$$|<T\psi_t^x, \phi_t^y>| = \left|\frac{1}{2}\iint K(z,u)\,(\psi_t^x(u)\,\phi_t^y(z) - \psi_t^x(z)\,\phi_t^y(u))\, dz\, du\right|$$

but $|\psi_t^x(u)\,\phi_t^y(z) - \phi_t^y(u)\,\psi_t^x(z)| \le \frac{|u-z|}{t^3}$ and the fact that $|u-x| < t$, $|u-z| < t$, $|x-y| < 3t$ and $|K(z,u)| \le \frac{1}{|z-u|}$ imply

$$|<T\psi_t^x, \phi_t^y>| \le \frac{C}{t} .$$

Combining these estimates proves our claim.

We can now state

THEOREM (G. David, J. L. Journe) [7]. *Let* $T:\mathcal{D} \to \mathcal{D}'$ *such that for some* $\varepsilon > 0$

$$|\langle T\psi_t^v, \phi_t^u \rangle| \le \frac{1}{t} \cdot \frac{1}{1 + \left|\frac{u-v}{t}\right|^{1+\varepsilon}} = p_t(u-v)$$

and

$$* \qquad |\langle T^*\psi_t^v, \phi_t^u \rangle| \le \frac{1}{t} \frac{1}{1 + \left|\frac{u-v}{t}\right|^{1+\varepsilon}} = p_t(u-v) .$$

Then the necessary and sufficient condition for T to extend to a bounded operator on L^2 into L^2 is that

$T(1)$ and $T^*(1)$ be in B.M.O.

We would like to make a few comments concerning the conditions $*$.

We have just seen that if

$$T(f) = \text{p.v.} \int k(x,y) f(y)\, dy$$

where

1° $$|k(x,y)| < \frac{1}{|x-y|}$$

2° $$|k(x,y^1) - k(x,y)| \le \frac{|y-y^1|^\varepsilon}{|x-y|^{1+\varepsilon}} \quad \text{for } |x-y| > 2|y-y^1|$$

and

$$|k(x^1,y) - k(x,y)| \le \frac{|x-x^1|^\varepsilon}{|x-y|^{1+\varepsilon}} \quad \text{for } |x-y| > 2|x-x^1|$$

for some $\varepsilon > 0$.

3° $$|\langle T(\psi_t^x), \phi_t^y \rangle| + |\langle T^*(\psi_t^x), \phi_t^y \rangle| \le \frac{c}{t} .$$