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STUDIES IN REAL AND COMPLEX ANALYSIS

I. I. Hirschman, Jr., editor

The Mathematical Association of America

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INTRODUCTION

I. I. Hirschman, Jr.

The eight articles in the present volume do not all presuppose the same mathematical background; they are directed generally to readers at the advanced undergraduate and first-year graduate level.

The initial article by H. J. Bremermann is a description of part of the modern theory of several complex variables which is centered about the successful efforts of mathematicians to understand fully the remarkable continuation properties possessed by analytic functions of several complex variables. Other topics central in this theory, such as the Cousin problems, analytic sets, etc., are discussed, although more briefly.

Graves' paper deals with a less extensive area, that of nonlinear functions from one Banach space to another, and in particular with the implicit function theorem. The material considered is treated in detail. Since this subject is beginning to make its way into advanced calculus texts, it is particularly fortunate to have this exposition. It is to be noted that Graves' paper has some elements in common with "Preliminaries to Functional Analysis" by Casper Goffman in Volume 1 of this series and that the two papers can be profitably read together.

Hille's paper on semi-groups gives a brief description of this vast area of analysis. The reader is introduced to such central, structural features of semigroup theory as the resolvant and the infinitesimal generator, and is also afforded a hint of the applications of this theory to stochastic processes and partial differential equations. Hille's article also makes contact with that of Goffman referred to above.

The article written by Hirschman and Widder is devoted to a relatively specific problem—the genesis of the real inversion formulas of the Laplace and Stieltjes transforms. These formulas and the corresponding representation theory are now seen, after some decades, to be a partly autonomous chapter within the very extensive area of totally positive matrices, variation diminishing transformations, and their extensions and generalizations.

Schaefer's paper is entirely different in spirit than the others in this volume in that it treats a classical subject, the Lebesgue-Stieltjes integral, in detail. Schaefer's approach is that of Daniell and F. Riesz; that is, the Lebesgue integral is constructed by an extension process from the Riemann integral, the theory of measure appearing only as a byproduct and at the end. Because it is both brief and rather complete, Schaefer's paper affords a unique opportunity to sample the elegance of this less familiar method. Moreover, this paper can serve as a convenient source for many of the measure theoretic results required in the other papers of these volumes.

Weiss' paper is simultaneously a detailed exposition of certain basic parts of harmonic analyses and an introduction to and description of selected advanced topics. The principal emphasis is on harmonic analysis in its classical form and here the exposition introduces the reader to the concept of "weak type" and to the Marcinkiewicz interpolation theorem, ideas which have played an important role in harmonic analysis in the last decade. The article concludes with a brief discussion of abstract harmonic analysis on locally compact Abelian groups.

Widom's paper is addressed to a rather specific problem, the inversion of semi-infinite Toeplitz operators. It can be profitably read in conjunction with Lorch's "The Spectral Theorem" in Volume 1 of this series. It is particularly interesting to see how, confronted with a concrete problem in spectral theory, the author draws on other phases of mathematics, in this case on the theory of Fourier series and analytic functions, in order to obtain a solution.

The articles of this volume treat only a small sample from the many topics of current interest in analysis, but it is believed that they are an interesting selection and it is hoped that the present volume will be a worthy successor to the elegant "Studies in Modern Analysis," which is Volume 1 in this series.

SEVERAL COMPLEX VARIABLES

H. J. Bremermann

The theory had its beginning shortly before the turn of the century. At first concepts and methods of the theory of one complex variable were generalized. Very soon, however, problems were encountered that were well understood in the case of one variable, but defied solution for two and more variables. Also, F. Hartogs [26], [27] (between 1906 and 1910) discovered profound results about analytic continuation and "natural boundaries" that are false for one variable. It became clear that the theory of several complex variables is not a mere generalization from one to n, but a distinct theory of its own.

After Hartogs, progress was slow for about twenty years. Then H. Behnke, H. Cartan, and P. Thullen developed the theory of domains and envelopes of holomorphy and S. Bergman began to investigate the kernel function and invariant metric (called after him).

In 1934 Behnke and Thullen summarized the knowledge up to that time in their book, *Theorie der Funktionen mehrerer komplexer Veränderlichen* [2] (still of interest).

Some of the outstanding problems mentioned in Behnke-Thullen have since been solved: (1) the analogue of "Runge's theorem," (2) construction of a meromorphic function to locally given poles (the so-called "additive Cousin problem") and construction of a holomorphic function to locally given zeros (multiplicative Cousin problem), and (3) the local characterization of the domains of holomorphy. The solutions of these problems are mostly due to K. Oka [32]–[40].

In recent years investigations have proceeded to complex manifolds and lately to "complex spaces," which are the n-dimensional analogues of Riemann surfaces. The language of "sheaves" has been developed and found to be an appropriate and powerful tool for studying functions and sets of functions on manifolds and complex spaces. Also, connections with Banach algebras have developed, and recently several complex variables have become important in theoretical physics (quantum field theory) [16].

Recently several books and notes on several complex variables have become available. B. A. Fuks [21] has appeared in a new and completely revised edition (translated into English) and a second volume has been added [22]. Topics that are of importance for quantum field theory have been treated by Vladimirov [50]. Excellent lecture notes have been compiled by L. Bers [7] and by L. Hörmander [28]. Of an earlier date are: Bochner-Martin [3] and Cartan seminaire 1951–1952 [19].

In what follows I will try to give a glimpse of the theory by emphasizing the problems mentioned above, around which much of the research has grown.

It is impossible in this limited space to deal with "sheaves," "complex manifolds," and "complex spaces." We can only touch on these subjects, giving references to the original literature. We also had to leave out Bergman's theory. An excellent introduction to this theory can be found in Bergman [5, chap. 11], and a more detailed representation in Bergman [6].

1. THE SPACE OF n-TUPLES OF COMPLEX NUMBERS C^n

From the familiar complex numbers we may form n-tuples. The collection of all n-tuples $z = (z_1, \cdots, z_n)$ of complex numbers $z_1, \cdots, z_n$ is denoted by C^n. We make it a linear vector space by introducing addition

$$z^{(1)} + z^{(2)} = (z_1^{(1)}, \cdots, z_n^{(1)}) + (z_1^{(2)}, \cdots, z_n^{(2)}) = (z_1^{(1)} + z_1^{(2)}, \cdots, z_n^{(1)} + z_n^{(2)});$$

and multiplication with a complex scalar λ

$$\lambda z = \lambda(z_1, \cdots, z_n) = (\lambda z_1, \cdots, \lambda z_n).$$

The addition is associative and commutative because it is defined as addition of the components, which are complex numbers. Analogously the multiplication by a scalar is distributive.

We leave it to the reader to verify that all the axioms of a linear vector space are satisfied.

1.1 The C^n becomes a Banach space by introducing a norm $\|\ \ \|$ satisfying: (1) $\|z\| > 0$ if $z \neq 0$; (2) $\|z^{(1)} + z^{(2)}\| \leq \|z^{(1)}\| + \|z^{(2)}\|$; (3) $\|\lambda z\| = (|\lambda|\,\|z\|)$, where λ is a complex number; (4) the C^n is complete with respect to the norm; that is, if for a sequence $\{z^{(j)}\}$, $z^{(j)} \in C^n$ we have $\|z^{(j)} - z^{(k)}\|$ tending to zero as j and k tend to infinity, then there exists an element $z^{(0)} \in C^n$ such that

$$\lim_{j\to\infty} \|z^{(j)} - z^{(0)}\| = 0.$$

Examples of Norms. The euclidean norm: $\|z\|_e^2 = |z_1|^2 + \cdots + |z_n|^2$. The maximum norm: $\|z\|_m = \max\{|z_1|, \cdots, |z_n|\}$. Every norm induces a topology if one defines as neighborhoods of a point $z^{(0)}$ the point sets

$$\{z \mid \|z - z^{(0)}\| < \epsilon;\ \epsilon > 0\}.$$

It is easy to show (the reader may carry out the proof) that: For any norm $\|\ \ \|$ there exist two numbers $\rho > 0$ and $\sigma > 0$ such that for any $z \in C^n$ we have

$$\rho\|z\|_m \leq \|z\| \leq \sigma\|z\|_m,$$

where $\|\ \ \|_m$ is the maximum norm.

A consequence of this fact is: *In the C^n all the topologies generated by different norms are equivalent.*

1.2 An *open set* C^n is called a *region*, and an *open and connected set* is called a *domain*.

1.3 The C^n is *topologically* and as *additive group* isomorphic to the additive group of $2n$-tuples of real numbers R^{2n} if we associate $z \longrightarrow (x_1, \cdots, x_n, y_1, \cdots, y_n)$, where $z_j = x_j + iy_j$ and $||(x, y)|| = ||z||$.

2. LINEAR SUBSPACES

2.1 We say that the C^n is of "complex dimension n." A *linear subspace of complex dimension* p is a subset of the C^n that can be written in the form

$$\{z \mid z = \lambda_1 a_1 + \cdots + \lambda_p a_p;\ (\lambda_1, \cdots, \lambda_p) \in C^p\},$$

where $a_1, \cdots, a_p \in C^n$ and fixed.

2.2 A "translated linear subspace"

$$\{z \mid z = z^{(0)} + \lambda_1 a_1 + \cdots + \lambda_p a_p;\ (\lambda_1, \cdots, \lambda_p) \in C^p\},$$

$(z^{(0)}, a_1, \cdots, a_p \in C^n$ and fixed) we will call a *complex p-plane*.

Instead of being defined by such a parameter representation, a complex p-plane can also be given by $n - p$ linear equations.

It should be noted that while every complex p-plane C^n is also a real $2p$-plane in the associated R^{2n}, the converse is not true. There are real $2p$-planes in the R^{2n} that are not complex p-planes in the C^n.

The reader may verify that the real 2-plane

$$E = \{z \mid x_1 = 0,\ x_2 = 0\}$$

is not a complex 1-plane in the C^2.

3. SPECIAL DOMAINS

An arbitrary domain in C^n can be visualized directly only for $n = 1$ because already for C^2 the associated real space is of real dimension 4. One method of visualizing domains D in C^2 is to fix one of the four associated real variables, for instance, y_2, and to look at the intersections

$$D \cap \{z \mid y_2 = y_2^{(0)}\}$$

for various values of $y_2^{(0)}$.

Some domains with sufficient symmetry can also be represented by a point set in a real space R^2 or R^3.

Examples

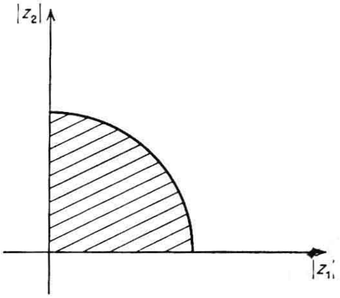

FIGURE 1

3.1 *The hypersphere:* $\{z \mid |z_1|^2 + \cdots + |z_n|^2 < r\}$. (This is the "ball" of radius r in the euclidean norm.) For $n = 2$ it can be represented graphically as shown in Fig. 1.

3.2 *The polycylinder:* $\{z \mid |z_1| < r_1, \cdots, |z_n| < r_n\}$. (For $r_1 = r_2 = \cdots = r_n = r$ this is the "ball" of radius r in the maximum norm.) The polycylinder (see Fig. 2) is the direct product of the n discs

$$\{z_1 \mid |z_1| < r_1\} \times \cdots \times \{z_n \mid |z_n| < r_n\}.$$

For complex dimension 1 both hypersphere and polycylinder coincide with the circle. For higher dimension they take with equal right the place of the circle, but they cannot even be mapped holomorphically onto each other. (This can be shown by means of invariants formed from Bergman's kernel function.)

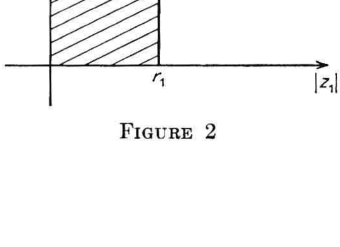

FIGURE 2

3.3 *Product domains:* $\{z \mid z_1 \in D_1, \cdots, z_n \in D_n\}$, where $D_1, \cdots, D_n$ are plane domains. See Fig. 3. The polycylinder is a product domain where the D_j are circles.

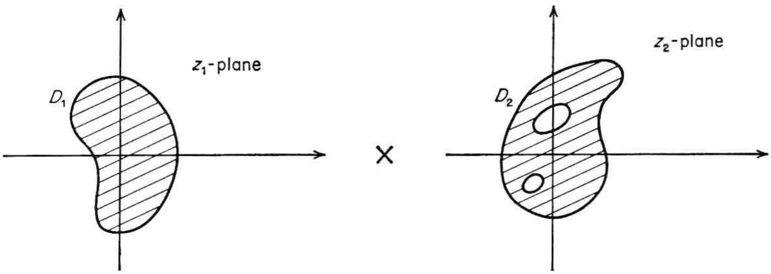

FIGURE 3

3.4 *Circular domains* (also denoted as "Reinhardt domains"):

$$\{z \mid (|z_1 - z_1^{(0)}|, \cdots, |z_n - z_n^{(0)}|) \in E\},$$

where E is a set in the $(|z_1 - z_1^{(0)}|, \cdots, |z_n - z_n^{(0)}|)$-space. A circular domain admits the automorphisms:

$$z_k^* - z_k^{(0)} = e^{i\theta_k}(z_k - z_k^{(0)}), \; k = 1, \cdots, n,$$

where $\theta_1, \cdots, \theta_n$ are arbitrarily real. See Fig. 4. The hypersphere and the polycylinder are special circular domains.

3.5 *Tube domains:* $\{z \mid x \in X, y \text{ arbitrary}\}$, $z_j = x_j + iy_j$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, and X is a domain in the space of the real parts $(x_1, \cdots, x_n)$. See Fig. 5.

3.6 *Hartogs domains:* $\{(z, w) \mid z \in D, r(z) < |w - w^{(0)}| < R(z)\}$, where D is a domain in the C^n, $w \in C^1$, and $r(z)$ and $R(z)$ are positive functions.

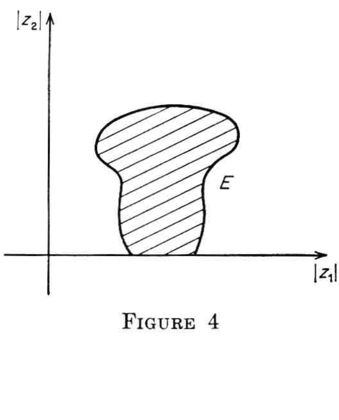

FIGURE 4

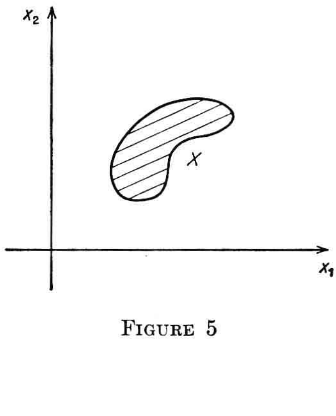

FIGURE 5

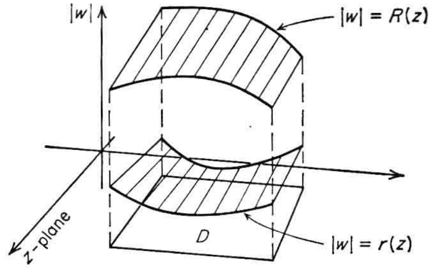

FIGURE 6

More generally, a Hartogs domain is a domain in (z, w)-space, $z \in C^n$, $w \in C^1$, that permits the following group of automorphisms: $z^* = z$, $w^* - w^{(0)} = e^{i\theta}(w - w^{(0)})$, θ arbitrary real. See Fig. 6.

4. HOLOMORPHIC FUNCTIONS

4.1 A function is the association of one and only one element in a certain "value set" to every element in an "argument set."

We will consider functions such that the values are complex (or real) and where the argument set is a domain in the C^n. p-tuples of such functions can be considered as one function with values in a C^p.

4.2 We remind the reader that the holomorphic functions of *one* complex variable can be characterized by four different properties:

A function $f(z)$ is holomorphic in a domain $D \subset C^1$ if and only if

(1) At each point $z^{(0)}$ of D it can be developed into a power series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z^{(0)})^\nu$$

that converges in a neighborhood of $z^{(0)}$.

(2) At each point $z^{(0)}$ of D the function $f(z)$ possesses a complex derivative. This is the case if and only if $f(z)$ possesses continuous

partial derivatives and the Cauchy-Riemann differential equations are satisfied in D. The latter can be written in the very simple form

$$\frac{\partial f}{\partial \bar{z}} = 0$$

if we introduce the differential operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In addition one also defines

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

(3) $w = f(z)$ maps the neighborhood of any point $z^{(0)} \in D$ at which $f'(z^{(0)}) \neq 0$ *conformally*. (That means: given two curves through z_0 and the angle between their tangents, then the angle between the tangents of the image curves in the w-plane is the same, in magnitude and direction.)

(4) $f(z)$ is holomorphic in D if and only if $f(z)$ is continuous in D and the integral $\int_{z^{(0)}}^{z} f(\zeta)\, d\zeta$; $z^{(0)}$, $z \in D$, is locally independent of the path of integration. (Cauchy's theorem and Morera's theorem.)

Each of these properties can be generalized to several variables and defines a class of functions. The question arises: are these classes of functions identical as they are for one variable?

4.3 Definition: *A complex-valued function $f(z)$ defined in a domain $D \subset C^n$ is called* holomorphic in D in the sense of Weierstrass *if it can be developed at each point $z^{(0)}$ of D into a multiple power series*

$$f(z) = \sum_{\nu_1, \ldots, \nu_n = 0}^{\infty} a_{\nu_1, \ldots, \nu_n}(z_1 - z_1^{(0)})^{\nu_1}, \cdots, (z_n - z_n^{(0)})^{\nu_n}$$

that converges uniformly in a neighborhood of $z^{(0)}$.

4.4 Definition: *A complex-valued function $f(z)$ defined in a domain $D \subset C^n$ is called* holomorphic in D in the sense of Cauchy-