Lecture Notes in Mathematics 1629

John Douglas Moore

# Lectures on Seiberg-Witten Invariants

Second Edition

Springer

John Douglas Moore

# Lectures on Seiberg-Witten Invariants

Second Edition

Springer

Author

John Douglas Moore
Department of Mathematics
University of California
Santa Barbara, CA 93106, USA

E-mail: moore@math.ucsb.edu

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Moore, John D.:
Lectures on Seiberg-Witten invariants / John Douglas Moore. - 2. ed..
- Berlin ; Heidelberg ; New York ; Barcelona ; Hong Kong ; London ;
Milan ; Paris ; Singapore ; Tokyo : Springer, 2001
(Lecture notes in mathematics ; 1629)
ISBN 3-540-41221-2

Mathematics Subject Classification (2000): 53C27, 58E15, 58J20

ISSN 0075-8434
ISBN 3-540-41221-2 Springer-Verlag Berlin Heidelberg New York



Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

http://www.springer.de



Typesetting: Camera-ready TEX output by the authors
SPIN: 10701161 41/3142-543210 - Printed on acid-free paper

# Preface

Riemannian, symplectic and complex geometry are often studied by means of solutions to systems of nonlinear differential equations, such as the equations of geodesics, minimal surfaces, pseudoholomorphic curves and Yang-Mills connections. For studying such equations, a new unified technology has been developed, involving analysis on infinite-dimensional manifolds.

A striking applications of the new technology is Donaldson's theory of "anti-self-dual" connections on $SU(2)$-bundles over four-manifolds, which applies the Yang-Mills equations from mathematical physics to shed light on the relationship between the classification of topological and smooth four-manifolds. This reverses the expected direction of application from topology to differential equations to mathematical physics. Even though the Yang-Mills equations are only mildly nonlinear, a prodigious amount of nonlinear analysis is necessary to fully understand the properties of the space of solutions.

At our present state of knowledge, understanding smooth structures on topological four-manifolds seems to require nonlinear as opposed to linear PDE's. It is therefore quite surprising that there is a set of PDE's which are even less nonlinear than the Yang-Mills equation, but can yield many of the most important results from Donaldson's theory. These are the Seiberg-Witten equations.

These lecture notes stem from a graduate course given at the University of California in Santa Barbara during the spring quarter of 1995. The objective was to make the Seiberg-Witten approach to Donaldson theory accessible to second-year graduate students who had already taken basic courses in differential geometry and algebraic topology.

In the meantime, more advanced expositions of Seiberg-Witten theory have appeared (notably [13] and [32]). It is hoped these notes will prepare the reader to understand the more advanced expositions and the excellent recent research literature.

We wish to thank the participants in the course, as well as Vincent Borrelli, Xianzhe Dai, Guofang Wei and Rick Ye for many helpful discussions on the material presented here.

J. D. MOORE
*Santa Barbara*
*April, 1996*

In the second edition, we have corrected several minor errors, and expanded several of the arguments to make them easier to follow. In particular, we included a new section on the Thom form, and provided a more detailed description of the second Stiefel-Whitney class and its relationship to the intersection form for four-manifolds. Even with these changes, the pace is demanding at times and increases throughout the text, particularly in the last chapter. The reader is encouraged to have pencil and paper handy to verify the calculations.

We have treated the Seiberg-Witten equations from the point of view of pure mathematics. The reader interested in the physical origins of the subject is encouraged to consult [9], especially the article, "Dynamics of quantum field theory," by Witten.

Our thanks go to David Bleecker for pointing out that our earlier proof of the Proposition on page 115 was incomplete, and to Lev Vertgeim and an anonymous referee for finding several misprints and minor errors in the text.

J. D. MOORE
*Santa Barbara*
*February, 2001*

# Contents

# Chapter 1

# Preliminaries

## 1.1 Introduction

During the 1980's, Simon Donaldson utilized the Yang-Mills equations, which had originated in mathematical physics, to study the differential topology of four-manifolds. Using work of Michael Freedman, he was able to prove theorems of the following type:

**Theorem A.** *There exist many compact topological four-manifolds which have no smooth structure.*

**Theorem B.** *There exist many pairs of compact simply connected smooth four-manifolds which are homeomorphic but not diffeomorphic.*

The nonlinearity of the Yang-Mills equations presented difficulties, so many new techniques within the theory of nonlinear partial differential equations had to be developed. Donaldson's theory was elegant and beautiful, but the detailed proofs were difficult for beginning students to master.

In the fall of 1994, Edward Witten proposed a set of equations which give the main results of Donaldson theory in a far simpler way than had been thought possible. The purpose of these notes is to provide an elementary introduction to the equations which Witten proposed. These equations are now known as the *Seiberg-Witten equations*.

Our goal is to use the Seiberg-Witten equations to give the differential geometric parts of the proofs of Theorems A and B. The basic idea is simple: one constructs new invariants of smooth four-manifolds, invariants which depend upon the differentiable structure, not just the topology.

The reader is undoubtedly familiar with many topological invariants of four-manifolds: the fundamental group $\pi_1(M)$, the cohomology groups

$H^k(M)$, the cup product, and so forth. These topological invariants have been around for a long time and have been intensively studied. The Seiberg-Witten equations give rise to new invariants of four-dimensional smooth manifolds, called the *Seiberg-Witten invariants*. The key point is that homeomorphic smooth four-manifolds may have quite different Seiberg-Witten invariants. Just as homology groups have many applications, one might expect the Seiberg-Witten invariants to have many applications to the geometry and differential topology of four-dimensional manifolds.

Indeed, shortly after the Seiberg-Witten invariants were discovered, several striking applications were found.

One application concerns the geometry of embedded algebraic curves in the complex projective plane $\mathbb{C}P^2$. Any such curve has a degree, which is simply the number of times the curve intersects a projective line in general position.

Algebraic topologists have another way of calculating the degree. A nonsingular algebraic curve can be regarded as the image of a holomorphic embedding

$$i : \Sigma \to \mathbb{C}P^2,$$

$\Sigma$ being a compact Riemann surface. The degree of the algebraic curve is the integer $d$ such that

$$i_*(\text{fundamental class in } H_2(\Sigma;\mathbb{Z})) = d \cdot (\text{generator of } H_2(\mathbb{C}P^2;\mathbb{Z})). \quad (1.1)$$

In many algebraic geometry texts (for example, page 220 in [19]), one can find a formula for the genus of an embedded algebraic curve:

$$g = \frac{(d-1)(d-2)}{2}.$$

Thom conjectured that if $\Sigma$ is a compact Riemann surface of genus $g$ and

$$i : \Sigma \to \mathbb{C}P^2$$

is any smooth embedding, not necessarily holomorphic, then

$$g \geq \frac{(d-1)(d-2)}{2},$$

the degree being defined by (1.1). (One would not expect equality for general embeddings, since one can always increase the genus in a fixed homology class by adding small handles.)

The Thom conjecture was proven by Kronheimer and Mrowka, Morgan, Szabo and Taubes, and Fintushel and Stern, using the Seiberg-Witten equations. These notes should give the reader adequate background to read the proof (versions of which are presented in [23] and [33]). The proof also gives much new information about embeddings of surfaces in four-manifolds other than $\mathbb{C}P^2$.

Another application of the Seiberg-Witten invariants comes from differential geometry. One of the most studied problems in Riemannian geometry concerns the relationship between curvature and topology of Riemannian manifolds. Perhaps the simplest type of curvature is the scalar curvature

$$s : M \to \mathbb{R}$$

of a Riemannian manifold $M$. The value of the scalar curvature at $p$ is a constant multiple of the average of all the sectional curvatures at $p$. It is interesting to ask: which compact simply connected Riemannian manifolds admit metrics with positive scalar curvature?

Lichnerowicz found the simplest obstruction to the existence of metrics of positive scalar curvature on compact simply connected manifolds. We will describe the part of Lichnerowicz's theorem that applies to four-manifolds later. Building upon the work of Lichnerowicz, Gromov and Lawson were able to obtain a relatively complete description of which compact simply connected manifolds of dimension $\geq 5$ admit metrics of positive scalar curvature. (See [25], Corollary 4.5, page 301.)

As Witten noticed, a compact four-manifold with positive scalar curvature must have vanishing Seiberg-Witten invariants. Thus there is an obstruction to the existence of metrics of positive scalar curvature which depends on the differentiable structure of the four-manifold, not just its topological type. The Seiberg-Witten invariants show that many compact four-manifolds (including all compact algebraic surfaces of "general type") do not admit metrics of positive scalar curvature.

A third application of the Seiberg-Witten equations is to symplectic geometry. Indeed, Taubes [38] was able to identify the Seiberg-Witten invariants of a compact symplectic four-manifold with Gromov invariants—as a consequence he obtained an existence theorem for "pseudoholomorphic curves" in such manifolds.

The rapidity with which these new results have been obtained suggests that the Seiberg-Witten equations may have yet further applications to the geometry of four-manifolds. This is now an area of intensive research.

The differential geometry needed to study the Seiberg-Witten equations is the geometry of spin and spin$^c$ structures. Until recently, these topics appeared unfamiliar and strange to many geometers, although spinors have long been regarded as important in physics. The tools needed to study spin and spin$^c$ structures are the same standard tools needed by all geometers and topologists: vector bundles, connections, characteristic classes and so forth. We will begin by reviewing some of this necessary background.

## 1.2 What is a vector bundle?

Roughly speaking, a vector bundle is a family of vector spaces, parametrized by a smooth manifold $M$.

How does one construct such a family of vector spaces? Suppose first that the ground field is the reals and the vector spaces are to be of dimension $m$, all isomorphic to $\mathbb{R}^m$. In this case, one starts with an open covering $\{U_\alpha : \alpha \in A\}$ of $M$ and for each $\alpha, \beta \in A$, smooth transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{R}) = \{m \times m \text{ nonsingular real matrices}\},$$

which satisfy the "cocycle condition"

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \qquad \text{on} \qquad U_\alpha \cap U_\beta \cap U_\gamma.$$

Note that

$$g_{\alpha\alpha} \cdot g_{\alpha\beta} = g_{\alpha\beta} \qquad \Rightarrow \qquad g_{\alpha\alpha} = I \quad \text{on} \quad U_\alpha,$$

and hence the cocycle condition implies that

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = g_{\alpha\alpha} = I \qquad \text{on} \qquad U_\alpha \cap U_\beta.$$

Let $\widetilde{E}$ denote the set of all triples $(\alpha, p, v) \in A \times M \times \mathbb{R}^m$ such that $p \in U_\alpha$. Define an equivalence relation $\sim$ on $\widetilde{E}$ by

$$(\alpha, p, v) \sim (\beta, q, w) \qquad \Leftrightarrow \qquad p = q \in U_\alpha \cap U_\beta, \quad v = g_{\alpha\beta}(p)w.$$

Denote the equivalence class of $(\alpha, p, v)$ by $[\alpha, p, v]$ and the set of equivalence classes by $E$. Define a projection map

$$\pi : E \to M \qquad \text{by} \qquad \pi([\alpha, p, v]) = p.$$

Let $\widetilde{U}_\alpha = \pi^{-1}(U_\alpha)$ and define a bijection

$$\psi_\alpha : \widetilde{U}_\alpha \to U_\alpha \times \mathbb{R}^m \qquad \text{by} \qquad \psi_\alpha([\alpha, p, v]) = (p, v).$$

There is a unique manifold structure on $E$ which makes each $\psi_\alpha$ into a diffeomorphism. With respect to this manifold structure, the projection $\pi$ is a smooth submersion.

A *real vector bundle of rank $m$ over $M$* is a pair $(E, \pi)$ constructed as above for some choice of open cover $\{U_\alpha : \alpha \in A\}$ of $M$ and some collection $g_{\alpha\beta}$ of transition functions which satisfy the cocycle condition. The *fiber* of this vector bundle over $p \in M$ is $E_p = \pi^{-1}(p)$, the preimage of $p$ under the projection. It has the structure of an $m$-dimensional real vector space.

When are two such vector bundles to be regarded as isomorphic? To answer this question, we need the notion of morphism within the category of vector bundles over $M$. A *vector bundle morphism* from $(E_1, \pi_1)$ to $(E_2, \pi_2)$ over $M$ is a smooth map $f : E_1 \to E_2$ which takes the fiber $(E_1)_p$ of $E_1$ over $p$ to the fiber $(E_2)_p$ of $E_2$ over $p$ and restricts to a linear map on fibers, $f_p : (E_1)_p \to (E_2)_p$. An invertible vector bundle morphism is called a *vector bundle isomorphism* over $M$. Let $\mathrm{Vect}^{\mathbb{R}}_m(M)$ denote the space of isomorphism classes of real vector bundles of rank $m$ over $M$.

The reader has no doubt encountered many examples of vector bundles in courses on differential geometry: the tangent bundle $TM$, the cotangent bundle $T^*M$, the $k$-th exterior power $\Lambda^k T^*M$ of the cotangent bundle, and other tensor bundles. Given two vector bundles $E_1$ and $E_2$ over $M$, one can form their direct sum $E_1 \oplus E_2$, their tensor product $E_1 \otimes E_2$, the bundle $\mathrm{Hom}(E_1, E_2)$, and so forth. One can also construct the dual bundle $E_1^*$ whose fibers are the dual spaces to the fibers of $E_1$. The construction of such vector bundles is described in detail in §3 of [30].

Complex vector bundles are defined in a similar way. The only difference is that in the complex case the transition functions take their values in the group $GL(m, \mathbb{C})$ of $m \times m$ complex matrices instead of $GL(m, \mathbb{R})$, and $\widetilde{E}$ is replaced by the set of triples $(\alpha, p, v) \in A \times M \times \mathbb{C}^m$ such that $p \in U_\alpha$. The construction described above then gives a pair $(E, \pi)$ in which the fiber $\pi^{-1}(p)$ has the structure of a *complex* vector space of dimension $m$. Let $\mathrm{Vect}^{\mathbb{C}}_m(M)$ denote the space of isomorphism classes of complex vector bundles of rank $m$ over $M$.

A complex vector bundle of rank one is also called a *complex line bundle*. The space of complex line bundles forms an abelian group under the tensor product operation $\otimes$. We will sometimes write

$$L^m = L \otimes L \otimes \cdots \otimes L \qquad (m \text{ times}).$$

Note that if

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(1, \mathbb{C})$$

are the transition functions for $L$, then the transition functions for $L^m$ are simply $g^m_{\alpha\beta}$.

In addition to real and complex vector bundles, one can define quaternionic vector bundles, vector bundles over the quaternions. Quaternions were first described by William R. Hamilton in 1853. In modern notation, a quaternion is simply a $2 \times 2$ matrix of the form

$$Q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $a$, $b$, $c$ and $d$ are real numbers, and

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is readily checked that the sum of two quaternions or the product of two quarternions is again a quaternion. Quaternion multiplication is bilinear over the reals; thus it is determined by the multiplication table for its basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$:

| | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
| $\mathbf{i}$ | $\mathbf{i}$ | $-\mathbf{1}$ | $-\mathbf{k}$ | $\mathbf{j}$ |
| $\mathbf{j}$ | $\mathbf{j}$ | $\mathbf{k}$ | $-\mathbf{1}$ | $-\mathbf{i}$ |
| $\mathbf{k}$ | $\mathbf{k}$ | $-\mathbf{j}$ | $\mathbf{i}$ | $-\mathbf{1}$ |

Thus of two possible conventions, we choose the one which induces the *negative* of the cross product on the three-plane of "imaginary quaternions" spanned by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Alternatively, we can think of quaternions as $2 \times 2$ complex matrices of the form

$$\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix},$$

where $z$ and $w$ are complex numbers. Note that since

$$\det Q = |z|^2 + |w|^2,$$

a nonzero quaternion $Q$ possesses a multiplicative inverse.

We let $\mathbb{H}$ denote the space of quaternions. It is a skew field, satisfying all the axioms of a field except for commutativity of multiplication. Let $GL(m, \mathbb{H})$ denote the group of nonsingular $m \times m$ matrices with quaternion entries.

To define a quaternionic vector bundle of rank $m$, we simply require that the transition functions $g_{\alpha\beta}$ take their values in $GL(m, \mathbb{H})$. We let $\text{Vect}_m^{\mathbb{H}}(M)$ denote the space of isomorphism classes of quaternionic vector bundles of rank $m$ over $M$.

Note that $GL(m, \mathbb{H})$ is a subgroup of $GL(2m, \mathbb{C})$, which in turn is a subgroup of $GL(4m, \mathbb{R})$. A quaternionic vector bundle of rank $m$ can thought of as a real vector bundle of rank $4m$ whose transition functions $g_{\alpha\beta}$ take their values in $GL(m, \mathbb{H}) \subset GL(4m, \mathbb{R})$. More generally, if $G$ is a Lie subgroup of $GL(m, \mathbb{R})$, a *$G$-vector bundle* is a rank $m$ vector bundle whose transition functions take their values in $G$.

Let us suppose, for example, that $G$ is the orthogonal group $O(m) \subset GL(m, \mathbb{R})$. In this case the transition functions of a $G$-vector bundle preserve the usual dot product on $\mathbb{R}^m$. Thus the bundle $E$ inherits a *fiber metric*, a smooth function which assigns to each $p \in M$ an inner product

$$\langle\ ,\ \rangle_p : E_p \times E_p \to \mathbb{R}.$$

If $G$ is the special orthogonal group $SO(n)$, a $G$-vector bundle possesses not only a fiber metric, but also an orientation.

Similarly, if $G$ is the unitary group $U(m) \subset GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$, a $G$-vector bundle is a complex vector bundle of rank $m$ together with a *Hermitian metric*, a smooth function which assigns to each $p \in M$ a map

$$\langle\ ,\ \rangle_p : E_p \times E_p \to \mathbb{C}$$

which satisfies the axioms

1. $\langle v, w \rangle_p$ is complex linear in $v$ and conjugate linear in $w$,
2. $\langle v, w \rangle_p = \overline{\langle w, v \rangle_p}$
3. $\langle v, v \rangle_p \geq 0$, with equality holding only if $v = 0$.

A *section* of a vector bundle $(E, \pi)$ is a smooth map

$$\sigma : M \to E \qquad \text{such that} \qquad \pi \circ \sigma = \text{identity}.$$

If $\sigma \in \Gamma(E)$, the restriction of $\sigma$ to $U_\alpha$ can be written in the form

$$\sigma(p) = [\alpha, p, \sigma_\alpha(p)], \qquad \text{where} \quad \sigma_\alpha : U_\alpha \to \begin{cases} \mathbb{R}^m \\ \mathbb{C}^m \\ \mathbb{H}^m \end{cases}$$

is a smooth map. The vector-valued functions $\sigma_\alpha$ are called the *local representatives* of $\sigma$ and they are related to each other by the formula

$$\sigma_\alpha = g_{\alpha\beta}\sigma_\beta \qquad \text{on} \quad U_\alpha \cap U_\beta. \tag{1.2}$$

In the real or complex case, the set $\Gamma(E)$ of sections of $E$ is a real or complex vector space respectively, and also a module over the space of smooth functions on $M$. In the quaternionic case, we need to be careful since quaternionic multiplication is not commutative. In this case, (1.2) shows that sections of $E$ can be multiplied on the right by quaternions.

**Example.** We consider complex line bundles over the Riemann sphere $S^2$, regarded as the one-point compactification of the complex numbers, $S^2 =$

$\mathbb{C}\cup\{\infty\}$. Give $\mathbb{C}$ the standard complex coordinate $z$ and let $U_0 = S^2 - \{\infty\}$, $U_\infty = S^2 - \{0\}$. For each integer $n$, define

$$g_{\infty 0} : U_\infty \cap U_0 \to GL(1, \mathbb{C}) \qquad \text{by} \qquad g_{\infty 0}(z) = \frac{1}{z^n}.$$

This choice of transition function defines a complex line bundle over $S^2$ which we denote by $H^n$. A section of $H^n$ is represented by maps

$$\sigma_0 : U_0 \to \mathbb{C}, \qquad \sigma_\infty : U_\infty \to \mathbb{C}$$

such that

$$\sigma_\infty = \frac{1}{z^n}\sigma_0, \qquad \text{on} \quad U_\infty \cap U_0.$$

It can be proven that any complex line bundle over $S^2$ is isomorphic to $H^n$ for some $n \in \mathbb{Z}$.

In particular, the cotangent bundle to $S^2$ must be isomorphic to $H^n$ for some choice of $n$. A section $\sigma$ of the cotangent bundle restricts to $\sigma_0 dz$ on $U_0$ for some choice of complex valued function $\sigma_0$. Over $U_\infty$, we can use the coordinate $w = 1/z$, and write $\sigma = -\sigma_\infty dw$. Since $dw = -(1/z)^2 dz$,

$$\sigma_0 dz = -\sigma_\infty dw \qquad \Rightarrow \qquad \sigma_\infty = z^2 \sigma_0,$$

and hence $n = -2$. In other words, $T^*S^2 = H^{-2}$. Similarly, $TS^2 = H^2$.

In a similar way, we can construct all quaternionic line bundles over $S^4$. In this case, we regard $S^4$ as the one-point compactification of the space of quaternions, $S^4 = \mathbb{H} \cup \{\infty\}$. Let $U_0 = \mathbb{H}$, $U_\infty = S^4 - \{0\}$, and define

$$g_{\infty 0} : U_\infty \cap U_0 \to GL(1, \mathbb{H}) \qquad \text{by} \qquad g_{\infty 0}(Q) = \frac{1}{Q^n}.$$

As $n$ ranges over the integers, we obtain all quaternionic line bundles over $S^4$.

How can we prove the claims made in the preceding paragraphs? Proofs can be based upon theorems from differential topology which classify vector bundles over manifolds. Here are two of the key results:

**Classification Theorem for Complex Line Bundles.** *If $M$ is a smooth manifold, there is a bijection*

$$\mathrm{Vect}_1^{\mathbb{C}}(M) \cong H^2(M; \mathbb{Z}).$$

This theorem will be proven in §1.6. The theorem implies that

$$\mathrm{Vect}_1^{\mathbb{C}}(S^2) \cong H^2(S^2; \mathbb{Z}) \cong \mathbb{Z},$$

and we will see that $H^m$ corresponds to $m \in \mathbb{Z}$ under the isomorphism. A argument similar to that for complex line bundles could be used to prove:

**Classification Theorem for Quaternionic Line Bundles.** *If $M$ is a smooth manifold of dimension $\leq 4$, there is a bijection*

$$\mathrm{Vect}_1^{\mathbb{H}}(M) \cong H^4(M;\mathbb{Z}).$$

## 1.3 What is a connection?

In contrast to differential topology, differential geometry is concerned with "geometric structures" on manifolds and vector bundles. One such structure is a connection. Evidence of the importance of connections is provided by the numerous definitions of connection which have been proposed.

A definition frequently used by differential geometers goes like this. Let

$$\chi(M) = \{\text{vector fields on } M\}, \qquad \Gamma(E) = \{\text{smooth sections of } E\}.$$

**Definition 1.** A *connection* on a vector bundle $E$ is a map

$$\nabla^A : \chi(M) \times \Gamma(E) \to \Gamma(E)$$

which satisfies the following axioms (where $\nabla^A_X \sigma = \nabla^A(X,\sigma)$):

$$\nabla^A_X(f\sigma + \tau) = (Xf)\sigma + f\nabla^A_X\sigma + \nabla^A_X\tau, \tag{1.3}$$

$$\nabla^A_{fX+Y}\sigma = f\nabla^A_X\sigma + \nabla^A_Y\sigma. \tag{1.4}$$

Here $f$ is a real-valued function if $E$ is a real vector bundle, a complex-valued function if $E$ is a complex vector bundle.

It is customary to regard $\nabla^A_X\sigma$ as the *covariant derivative* of $\sigma$ in the direction of $X$.

Given a connection $\nabla^A$ in the sense of Definition 1, we can define a map

$$d_A : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Gamma(\mathrm{Hom}(TM, E))$$

by

$$d_A(\sigma)(X) = \nabla^A_X\sigma.$$

Then $d_A$ satisfies a second definition:

**Definition 2.** A *connection* on a vector bundle $E$ is a map

$$d_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

which satisfies the following axiom:

$$d_A(f\sigma + \tau) = (df) \otimes \sigma + f d_A \sigma + d_A \tau. \tag{1.5}$$

Definition 2 is more frequently used in gauge theory, but in our presentation both definitions will be important. Note that Definition 2 is a little more economical in that one need only remember one axiom instead of two. Moreover, Definition 2 makes clear the analogy between a connection and the exterior derivative.

The simplest example of a connection occurs on the bundle $E = M \times \mathbb{R}^m$, the trivial real vector bundle of rank $m$ over $M$. A section of this bundle can be identified with a vector-valued map

$$\sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \cdot \\ \sigma^m \end{pmatrix} : M \to \mathbb{R}^m.$$

We can use the exterior derivative to define the "trivial" flat connection $d_A$ on $E$:

$$d_A \begin{pmatrix} \sigma^1 \\ \cdot \\ \sigma^m \end{pmatrix} = \begin{pmatrix} d\sigma^1 \\ \cdot \\ d\sigma^m \end{pmatrix}.$$

More generally, given an $m \times m$ matrix

$$\omega = \begin{pmatrix} \omega_1^1 & \cdot & \omega_m^1 \\ \cdot & \cdots & \cdot \\ \omega_1^m & \cdot & \omega_m^m \end{pmatrix}$$

of real-valued one-forms, we can define a connection $d_A$ by

$$d_A \begin{pmatrix} \sigma^1 \\ \cdot \\ \sigma^m \end{pmatrix} = \begin{pmatrix} d\sigma^1 \\ \cdot \\ d\sigma^m \end{pmatrix} + \begin{pmatrix} \omega_1^1 & \cdot & \omega_m^1 \\ \cdot & \cdots & \cdot \\ \omega_1^m & \cdot & \omega_m^m \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \cdot \\ \sigma^m \end{pmatrix}. \tag{1.6}$$

We can write this last equation in a more abbreviated fashion:

$$d_A \sigma = d\sigma + \omega\sigma,$$

matrix multiplication being understood in the last term. Indeed, the axiom (1.5) can be verified directly, using the familiar properties of the exterior derivative:

$$d_A(f\sigma + \tau) = d(f\sigma + \tau) + \omega(f\sigma + \tau)$$