UNIVERSITY MATHEMATICS

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UNIVERSITY MATHEMATICS II

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Preface

This is the second of two volumes that are intended to provide college and university students with a sensible continuation of the modern approach to mathematics that is being introduced in most elementary and secondary schools, with more emphasis than in the past placed on an understanding of fundamental concepts. Certain advanced topics in algebra and trigonometry, along with analytic geometry and calculus, are unified into a sequential exposition that eliminates much unnecessary duplication and is conducive to an efficient development and use of ideas and techniques. Fundamental concepts are discussed in a reasonably rigorous fashion, with adequate emphasis on important skills, and without an excess of sophistication. Many applications of mathematics have been included, and they have frequently been made the motivation for the introduction of mathematical concepts. An intuitive discussion often precedes the formal treatment of a new idea.

Although the books were written with students in engineering and the sciences in mind, they are also well suited for a good liberal arts course in mathematics. The exposition has, in the main, been kept at a level that has proved to be reasonable for the average student. However, a number of optional sections, problems, and proofs, each of which is marked by a star and may be omitted without loss of continuity, have been included as a challenge to the better students.

Important definitions, axioms, and theorems are clearly labeled, and a conscientious effort has been made to utilize each new idea and notation as frequently as possible in order to promote its intelligent use by the student. New materials and new points of view are not introduced merely for the sake of novelty, but are brought in only if they make a genuine contribution to the understanding that can be imparted to the reader.

There are several features of particular interest that we have found helpful in providing the student with a deeper understanding of elementary mathematical analysis, as well as a better background for mathematics beyond the sophomore level. First, there is the development and consistent use of the neighborhood concept in the treatment of limits. This approach gives the student a better intuitive feeling for the meaning of a limit than the more usual formal ϵ-δ attack. The second significant feature is the introduction and use of matrices for the solution of systems of linear equations and for the discussion of linear transformations in reducing a quadratic polynomial to a canonical form, as well as the application of these ideas to the solution of simple systems of differential equations. A third important feature is the use of vector algebra for the discussion of geometric ideas relating to the line and the plane in three-dimensional space, and the use of vector calculus for the development of a number of basic notions relating to curves and surfaces as well as to velocity and acceleration. The introduction and application of some elementary ideas in the calculus of complex-valued functions motivates and simplifies the use of the exponential function with an imaginary exponent.

The material in these books has been taught quite successfully for the past three years—first in the form of notes and then in an offset preliminary edition—to ordinary freshman and sophomore classes. The point of view of the exposition, the organization, and the development of the mathematical ideas, the new topics, and the intuitive development that often precedes a more rigorous formal discussion, have all been enthusiastically received by both faculty and students. We believe that this approach has enabled students to attain a desirable level of mathematical maturity in a shorter time than they could have with the more traditional approaches.

The first seven chapters of Volume I are concerned with basic ideas and the development of a consistent language and terminology for the remainder of the book. A good modern course in analytic geometry and calculus can be based on Chapters 4 and 5, the first three sections of Chapter 6, and Chapters 8 to 15 of Volume I, plus Chapters 1 to 11 of Volume II. Chapters 12 and 13 of Volume II contain adequate material for a short course in differential equations. Chapter 14 consists of an elementary treatment of the Laplace transformation, and Chapter 15 is a brief introduction to probability.

The material in Volume I can easily be covered in two five-semester-hour courses in the freshman year. The material in the first thirteen of the fifteen chapters of Volume II can be covered (with minor omissions) in two four-semester-hour courses in the sophomore year. It is, however, quite possible for a well-prepared class to complete both volumes in the two-year sequence by omitting the more elementary portions of Volume I. In order to establish the language and point of view for such students, it is advisable to study the concept of a set and the set notation in Sections 1.6, 1.7, and 1.8. The summary of Chapter 2 gives the symbols that are consistently used to denote certain special sets of numbers. Basic work on inequalities occurs in Sections 3.8 and 3.9. Chapters 4, 5, and 6, which contain the introductory work in analytic geometry and the discussion of relations and functions, should be taken in more or less detail, depending on the preparation of the class. Chapter 7, which is concerned with basic trigonometry, may be omitted for students with good high school preparation in this subject. Not more than two or three weeks is needed to cover the preceding topics, so that students with adequate high school background are then able to begin the serious work on limits and continuity in Chapter 8.

We wish to thank Professors R. A. Rosenbaum and Morris Kline for their editorial suggestions, which contributed in a notable fashion to the clarity of the exposition. Many other valuable suggestions came from our colleagues in the Department of Applied Mathematics at the University of Colorado and from the long-suffering students who have seen the book through the many pains of its birth. To these students and colleagues we owe a debt that can be repaid only by the gratitude of a newer generation of students for whom the exposition has been made simpler and clearer. We are particularly appreciative of the intelligent effort put in by Mrs. Dorothy Vaughn in typing the manuscript in its many revisions. Finally, we wish to express our gratitude to our families for putting up with us during the trials and tribulations of this project.

March 1965

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R. BEN KRIEGH
LEON W. RUTLAND

Contents

Chapter 1 Coordinate Geometry

1.1 SETS OF POINTS DETERMINED BY GEOMETRIC CONDITIONS

The first organization of geometric concepts into an axiomatic system was given by Euclid about 300 B.C. Euclid's geometry was a study of geometric figures based on certain axioms suggested by physical considerations. In the early 1600's, the French mathematician René Descartes (1596–1650) introduced the concept of a coordinate system by means of which he was able to translate geometric problems into an algebraic language. The resulting union of algebra and geometry, now known as **analytic** or **coordinate geometry,** proved to be so powerful that it opened the door to an amazing new era of mathematics.

The fundamental ideas of coordinate geometry, such as the representation of points in a plane by ordered pairs of real numbers and the representation of a curve by an algebraic equation, were introduced in Chapter 4 of Volume I. By means of these representations, we were able to study the straight line in considerable detail and, in particular, we saw how to specify the direction of a straight line by means of the slope or by means of direction numbers. We were also able to analyze properties of curves, to find symmetries, asymptotes, and so on; and with the aid of the calculus, to describe other salient features of a curve. The importance and value of these ideas in the analysis of many physical problems can hardly be overestimated.

In order to see further how these concepts are used, we shall investigate additional simple applications to geometric problems. The set of all ordered pairs of real numbers is called a **space,** and a particular ordered pair in the set is called a **point** in the space. The distance between two points (x_1, y_1) and (x_2, y_2) was defined in Chapter 4 of Volume I as

$$d = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}.$$

The expression for d is called the **metric** of the space, and the space is called a **euclidean metric space** of two dimensions. This space is the familiar xy-plane.

Suppose we wish to describe analytically a set of points each of which is located twice as far from the x-axis as from the y-axis. If the distance of an arbitrary point of the set from the y-axis is denoted by d_1, then the distance of the point from the x-axis is $2d_1$ (see Figure 1.1a). In analytic form, we have

$$d_1 = |x| \quad \text{and} \quad 2d_1 = |y|,$$

so that the equations

$$2x = y \quad \text{and} \quad 2x = -y$$

describe the set of points, since the coordinates of each point in the set satisfy one or the other of these equations.

The next example illustrates another simple geometric problem.

Example 1.1a. Find an analytic description for the set of points located so that the distance of each from the point (1, 0) is equal to its distance from the line $x = 0$.

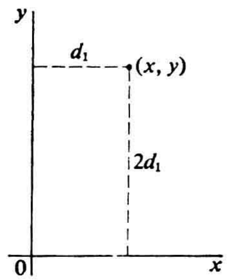

FIGURE 1.1a

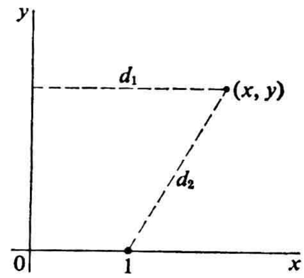

FIGURE 1.1b

As Figure 1.1b shows, d_1 represents the distance of the point (x, y) from the y-axis, and d_2 its distance from the point (1, 0). The condition to be satisfied is that $d_1 = d_2$. From

$$d_1 = |x| \quad \text{and} \quad d_2 = [(x - 1)^2 + y^2]^{1/2},$$

we get

$$|x| = [(x - 1)^2 + y^2]^{1/2},$$
$$x^2 = (x - 1)^2 + y^2.$$

This equation reduces to

$$y^2 = 2x - 1.$$

In other words, every point (x, y) that satisfies the given geometric condition must have coordinates that satisfy the above algebraic condition. But can we be sure that every point whose coordinates satisfy the equation also satisfies the geometric condition? In this case, the desired assurance can be obtained by working backward from the final equation to obtain the result that $d_1 = d_2$. Thus, if

$$y^2 = 2x - 1,$$

then

$$y^2 + x^2 - 2x + 1 = x^2,$$

or

$$y^2 + (x - 1)^2 = x^2.$$

Since the left side of this equation is the square of the distance d_2 of the point (x, y) from the point (1, 0), and the right side is the square of the distance d_1 of the point from the y-axis, we see that $d_1 = d_2$ as required.

The next example shows that it is sometimes necessary to adjoin an inequality to the equation in order to describe the geometric conditions completely.

Example 1.1b. Find an equation to describe the set of points $\{(x, y)\}$ in the first quadrant such that the product of the coordinates of each point is always 6.

It is easy to write that $xy = 6$. Unfortunately, this equation is satisfied by the coordinates of points in the third quadrant, such as $(-2, -3)$, which we wish to exclude. Hence, the equation $xy = 6$ alone is not adequate. Instead we must write

$$xy = 6, \qquad x > 0$$

to describe the given set of points.

Example 1.1c. Find an equation for the set of all points located at a distance of 5 units from the point (3, 4).

Let $P(x, y)$ be a typical point of the set (see Figure 1.1c). The distance of this point from (3, 4) is $d = [(x - 3)^2 + (y - 4)^2]^{1/2}$, which must equal 5. Hence,

$$5 = [(x - 3)^2 + (y - 4)^2]^{1/2}$$

or

$$x^2 - 6x + y^2 - 8y = 0.$$

Example 1.1d. Find an equation for the set of points such that the product of the slopes of the two lines joining each point of the set to the points (0, 2) and (0, −2) is 1.

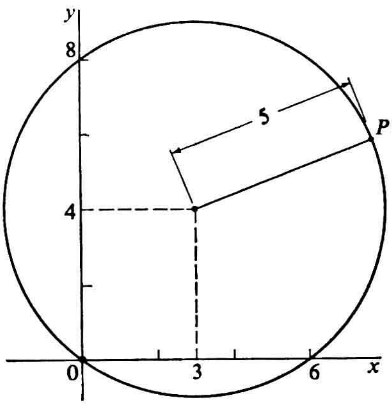

FIGURE 1.1c

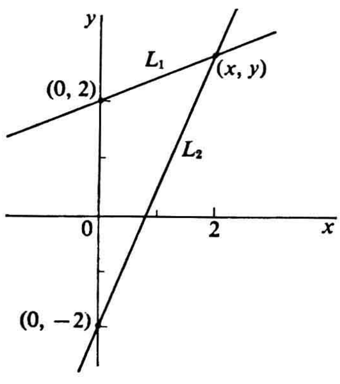

FIGURE 1.1d

Referring to Figure 1.1d, where (x, y) is supposed to be a point in the given set, we have, for the slope of L_1,

$$m_1 = \frac{y - 2}{x}, \qquad x \neq 0,$$

and, for the slope of L_2,

$$m_2 = \frac{y + 2}{x}, \qquad x \neq 0.$$

Since the given condition is $m_1m_2 = 1$, we get

$$\left(\frac{y-2}{x}\right)\left(\frac{y+2}{x}\right) = 1,$$

or

$$y^2 - x^2 = 4, \qquad x \neq 0.$$

The final equation is that of a hyperbola with its vertices at (0, 2) and (0, −2), but the vertices themselves are not in the given set of points.

The next example illustrates another technique that is helpful in determining equations for sets of points.

Example 1.1e. Find an equation for the set of midpoints of the chords drawn from the origin to the points of the curve determined by $y^2 = x$.

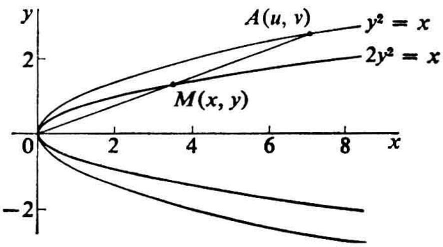

FIGURE 1.1e

Figure 1.1e shows the graph of $y^2 = x$ and a typical chord. Let the point $A(u, v)$ be a point of the curve $y^2 = x$, and let the point $M(x, y)$ be the midpoint of the chord OA. The coordinates of the point A on the given curve are denoted by u and v in order to distinguish them from the coordinates of a point $M(x, y)$ on the required curve.

It is now necessary to find a connection between the coordinates (u, v) and (x, y). In this case the desired relationship is easily obtained, since it is known that M is the midpoint of the chord. Hence,

$$x = \tfrac{1}{2}u \quad \text{and} \quad y = \tfrac{1}{2}v,$$

or

$$2x = u, \quad 2y = v.$$

Since u and v satisfy the equation $v^2 = u$, we get

$$(2y)^2 = 2x,$$

or

$$2y^2 = x$$

as an equation of the set of points.

The preceding example illustrates a device that is frequently convenient in analytic geometry—namely, giving a point on an unknown curve the general coordinates (x, y) and then finding a relationship between these coordinates and the coordinates of a point (u, v) on a known curve.

Example 1.1f. Find an equation for the set of all points located 4 units farther from (4,0) than from the line $x = 2$.

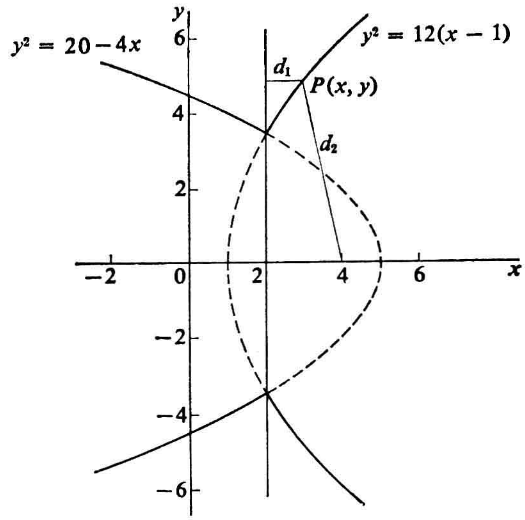

FIGURE 1.1f

If $P(x, y)$ is a typical element of the set (Figure 1.1f), then the distances of P from the line and from the point are, respectively,

$$d_1 = \sqrt{(x-2)^2} \quad \text{and} \quad d_2 = \sqrt{(x-4)^2 + y^2}.$$

The problem states $d_2 = d_1 + 4$. However, $d_1 \geqq 0$ and $d_2 \geqq 0$, so that care must be used in performing algebraic operations on this equation. Keep in mind that the distances may not be negative and that we must use

$$\sqrt{(x-2)^2} = \begin{cases} x - 2 & \text{for } x \geqq 2, \\ 2 - x & \text{for } x < 2. \end{cases}$$

Thus, the problem must be considered in two parts. For $x \geqq 2$, we have

$$\begin{aligned} x - 2 + 4 &= \sqrt{(x-4)^2 + y^2}, \\ (x+2)^2 &= (x-4)^2 + y^2, \\ y^2 &= 12(x-1). \end{aligned}$$

For $x < 2$, we have

$$\begin{aligned} 2 - x + 4 &= \sqrt{(x-4)^2 + y^2}, \\ (6-x)^2 &= (x-4)^2 + y^2, \\ y^2 &= 20 - 4x. \end{aligned}$$

Accordingly, we may write for the required set of points:

$$y^2 = \begin{cases} 12(x-1), & x \geqq 2, \\ 4(5-x), & x < 2. \end{cases}$$

The graph consists of the heavy solid lines in Figure 1.1f.