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ALGORITHMIC ASPECTS OF GRAPH CONNECTIVITY

Hiroshi Nagamochi
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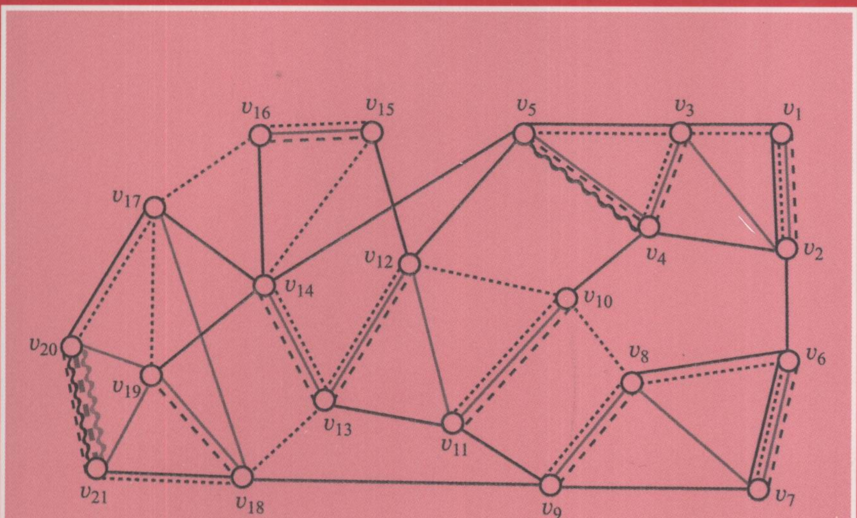

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Algorithmic Aspects of Graph Connectivity

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Algorithmic Aspects of Graph Connectivity

Algorithmic Aspects of Graph Connectivity is the first book that thoroughly discusses graph connectivity, a central notion in graph and network theory, emphasizing its algorithmic aspects. This book contains various definitions of connectivity, including edge-connectivity, vertex-connectivity, and their ramifications, as well as related topics such as flows and cuts. With wide applications in the fields of communication, transportation, and production, graph connectivity has made tremendous algorithmic progress under the influence of theory of complexity and algorithms in modern computer science. New concepts and graph theory algorithms that provide quicker and more efficient computing, such as MA (maximum adjacency) ordering of vertices, are comprehensively discussed.

Covering both basic definitions and advanced topics, this book can be used as a textbook in graduate courses of mathematical sciences (such as discrete mathematics, combinatorics, and operations research) in addition to being an important reference book for all specialists working in discrete mathematics and its applications.

Hiroshi Nagamochi is a professor at the Graduate School of Informatics, Kyoto University. He is a member of the Operations Research Society of Japan and the Information Processing Society.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

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Preface

Because the concept of a graph was introduced to represent how objects are connected, it is not surprising that connectivity has been a central notion in graph theory since its birth in the 18th century. Various definitions of connectivities have been proposed, for example, edge-connectivity, vertex-connectivity, and their ramifications. Closely related to connectivity are flows and cuts in graphs, where the cut may be regarded as a dual concept of connectivity and flows.

A recent general trend in the research of graph theory appears as a shift to its algorithmic aspects, and improving time and space complexities has been a strong incentive for devising new algorithms. This is also true for topics related to connectivities, flows, and cuts, and much important progress has been made. Such topics include computation, enumeration, and representation of all minimum cuts and small cuts; new algorithms to augment connectivity of a given graph; their generalization to more abstract mathematical systems; and so forth. In view of these, it would be a timely attempt to summarize those results and present them in a unified setting so that they can be systematically understood and can be applied to other related fields.

In these developments, we observe that a simple tool known as maximum adjacency (MA) ordering has been a profound influence on the computational complexity of algorithms for a number of problems. It is defined as follows.

> MA ordering: Given a graph $G = (V, E)$, a total ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of vertices is an MA ordering if $|E(V_{i-1}, v_i)| \geq |E(V_{i-1}, v_j)|$ holds for all i, j with $2 \leq i < j \leq n$, where $V_i = \{v_1, v_2, \ldots, v_i\}$ and $E(V', v)$ is the set of edges from vertices in V' to v.

To our knowledge, MA ordering was first introduced in a paper by R. E. Tarjan and M. Yannakakis [300], where it was called the Maximum Cardinality Search and used to test chordality of graphs, to test acyclicity of hypergraphs, and to solve other problems. We then rediscovered MA ordering [232], showing that it is effective for problems such as finding a forest decomposition and computing the minimum cuts of a graph. The extension in this direction has continued, and many problems are found to have faster algorithms.

The topics covered in this book are forest decomposition, minimum cuts, small cuts, cactus representation of cuts, connectivity augmentation, and source location problems. Mathematical tools used to solve these problems, such as maximum flows, extreme vertex sets, and edge splitting, are also discussed in detail. A generalization to a more abstract system than a graph is attempted on the basis of submodular and posimodular set functions.

The primary purpose of this book is to serve as a research monograph that covers the aforementioned algorithmic results attained in the area of graph connectivity, putting emphasis on results obtained from the introduction of MA ordering. However, this book is also appropriate as a textbook in graduate courses of mathematical sciences and operations research, because it starts with basic definitions of graph theory and contains most of the important results related to graph connectivities, flows, and cuts. Because the concept of connectivity is an important notion in many application areas, such as communication, transportation, production, scheduling, and power engineering, this book can be used as a reference for specialists working in such areas.

We would like to express our deep thanks to the many people who helped us to complete this project. First of all, we appreciate all the collaborations and comments given to us by Peter Eades, Andras Frank, Satoru Fujishige, Takuro Fukunaga, Magnús M. Halldórsson, Seokhee Hong, Toshimasa Ishii, Satoru Iwata, Tibor Jordán, Yoko Kamidoi, Kazuhisa Makino, Kiyohito Nagano, Mariko Sakashita, Kei Yamashita, and Liang Zhao, among others. We are particularly grateful to the late Professor Peter Hammer of Rutgers University for encouraging us to write this book. Finally we extend our thanks to our wives, Yuko and Mizuko, respectively, for their generous understanding.

Hiroshi Nagamochi
Toshihide Ibaraki
2007

Notation

$\Re$	set of reals	1
$\Re_+$	set of nonnegative reals	1
$\Re_-$	set of nonpositive reals	1
$\mathbf{Z}$	set of integers	1
$\mathbf{Z}_+$	set of nonnegative integers	1
$\mathbf{Z}_-$	set of nonpositive integers	1
$\lceil a \rceil$	smallest integer not smaller than a	1
$\lfloor a \rfloor$	largest integer not larger than a	1
$[a, b]$	closed interval; set of reals c with $a \leq c \leq b$	1
(a, b)	open interval; set of reals c with $a < c < b$	1
$\|V\|$	cardinality of a set V	1
2^V	power set of V	1
$\binom{V}{2}$	set of all pairs of elements in V	1
$V(G)$	vertex set of a graph G	2
$E(G)$	edge set of a graph G	2
n	$\|V\|$	2
m	$\|E\|$	2
$V[F]$	set of end vertices of edges in F	2
$h(e)$	the head of a directed edge e	2
$t(e)$	the tail of a directed edge e	2
$\delta(G)$	minimum degree of a graph G	2
$\Delta(G)$	maximum degree of a graph G	2
$c_G(e)$	weight of edge e in G	3
$c_G(u, v)$	weight of edge $\{u, v\}$ in G	3
$E(X, Y; G)$	set of undirected edges joining a vertex in X and a vertex in Y for undirected graph G; set of directed edges with a tail in X and a head in Y for directed graph G	4
$d(X, Y; G)$	$\sum_{e \in E(X,Y;G)} c_G(e)$	3
$E(X; G)$	$E(X, V - X; G)$	4

$d(X; G)$	$d(X, V - X; G)$ for undirected graph G, where $d(\emptyset; G) = d(V; G) = 0$ is assumed	4
$d^+(X; G)$	$d(X, V - X; G)$ for directed graph G	4
$d^-(X; G)$	$d(V - X, X; G)$ for directed graph G	4
$\Gamma_G(v)$	set of neighbors of v in G	6
$\Gamma_G^+(X)$	set of out-neighbors of v in G	6
$\Gamma_G^-(X)$	set of in-neighbors of v in G	6
$G - F$	graph obtained from G by removing edges in F	7
G/F	graph obtained from G by contracting each edge in F into a single vertex and deleting any resulting loops	7
$G + E'$	graph obtained from G by adding the edges in E'	8
$G[X]$	subgraph induced from G by X	8
$G - X$	graph obtained from G by removing the vertices in X together with the edges incident with a vertex in X	8
G/X	graph obtained from G by contracting vertices in X into a single vertex and deleting any resulting loops	8
$G + b$	star augmentation of G defined by b	8
$\lambda(u, v; G)$	local edge-connectivity between u and v	9
$\lambda(S, v; G)$	size of a cut separating S and v	10
$\lambda(G)$	edge-connectivity of G	10
$\kappa(G)$	vertex-connectivity of G	10
$\kappa(u, v; G)$	local vertex-connectivity between u and v	10
$\kappa(S, v; G)$	minimum size of a vertex cut separating S and v	11
$\hat{\kappa}(S, v; G)$	maximum number of paths between S and v such that no two paths share any vertex other than v	11
e^r	reversal edge of e	22
$dist(u, v; G)$	distance from u to v in G	26
$\hat{G}$	digraph obtained by contracting all the strongly connected components in G	31
$\psi_G(v)$	$\sum\{c_G(e) \mid e = (v, u) \in E\} - \sum\{c_G(e) \mid e = (u, v) \in E\}$	33
$\lambda_\alpha(u, v; G)$	local α-connectivity	36
$\lambda_T(u, v; G)$	local T-connectivity	37
$\mu_\ell(u, v; G)$	local ℓ-mixed connectivity	37
$\lambda_s^+(G)$	$\min\{\lambda(s, v; G) \mid v \in V - s\}$	38
$\lambda_s^-(G)$	$\min\{\lambda(v, s; G) \mid v \in V - s\}$	38
$\overline{E}$	set of ordered pairs (u, v) such that $u, v \in V$, $u \neq v$ and $(u, v) \notin E$	39
$\overline{E}(X, Y)$	$\{(u, v) \in \overline{E} \mid u \in X, v \in Y\}$	39
$\kappa_s^+(G)$	$\min\{\kappa(s, v; G) \mid (s, v) \in \overline{E}(s, V - s)\}$	39
$\kappa_s^-(G)$	$\min\{\kappa(v, s; G) \mid (v, s) \in \overline{E}(V - s, s)\}$	39
G_S	digraph obtained by adding to G a new vertex s and directed edges (s, v) and (v, s) for every $v \in S$	40
$\kappa_{s,T}(G_S)$	$\min\{\kappa(s, v; G_S) \mid (s, v) \in \overline{E}(s, T; G_S)\}$	41

$\kappa_{T,s}(G_S)$	$\min\{\kappa(v, s; G_S) \mid (v, s) \in \overline{E}(T, s; G_S)\}$	41
$\alpha(n, n)$	inverse function of Ackermann function	45
$\mathcal{Y}(G)$	set of all maximal components of G	51
$\mathcal{X}(G)$	family of all extreme vertex sets of G	53
$\mathcal{C}(\mathcal{R})$	set of all minimum cuts in $\mathcal{R}$	55
$\tau(\mathcal{E})$	transversal number of a hypergraph $\mathcal{E}$	60
$\nu(\mathcal{E})$	matching number of a hypergraph $\mathcal{E}$	61
$D(v)$	set of all descendants of v in a tree	61
v_X	a unique vertex in X such that $X \subseteq D(v_X)$	61
$L(\mathcal{E})$	line graph of a hypergraph $\mathcal{E}$	62
G_w	edge-weighted complete graph defined such that $c(u, v) = \sum_{X \in \mathcal{E}:\{u,v\} \subseteq X} w(X)$ for hyperedge weight w	64
$\mathcal{C}_k(G)$	set of cuts with size k in G	67
$\overline{G}$	digraph obtained from a digraph G by reversing the direction of every edge	69
$U(G)$	underlying graph of a digraph G	72
$\mathcal{C}_k(u, v; G)$	set of all mixed cuts having size k and separating vertices u and v in G	79
S_e	edge set such that $e \in S_e$ and $e' \in S_e$ if and only if $\{e, e'\}$ is a 2-cut	88
$G\downarrow \deg 2$	graph obtained by repeating the operation to delete $E(v, V - v; G)$ and to add new edge connecting the two neighbors of v for all vertices v with degree 2	93
$G\downarrow e$	$(V, (E - S_e) \cup D_e)$, where $S_e = \{\{x_0, y_0\}, \ldots, \{x_h, y_h\}\}$ and $D_e = \{\{y_0, x_1\}, \ldots, \{y_h, x_0\}\}$	93
$[X]_G$	for a subset of vertices in G/E', set of all vertices in V that are contracted to some vertices in X	95
$[M_3(G/E')]_G$	$\{[X_1]_G, [X_2]_G, \ldots, [X_p]_G\}$ for $M_3(G/E') = \{X_1, X_2, \ldots, X_p\}$	95
$G + a \times X$	graph obtained by adding vertex a and edge $\{a, u\}$ for every vertex $u \in X$	103
$G + X \times X$	graph obtained by adding edges $\{u, v\}$ for all nonadjacent pairs of vertices $u, v \in X$	103
$val(s, t; H)$	value of a maximum (s, t)-flow in an undirected graph or digraph H	108
$\widetilde{G}$	digraph obtained by replacing each edge with two oppositely oriented edges in an undirected graph G	108
G^f	residual digraph defined by $\widetilde{G}$ and (s, t)-flow f	108
$E^{f1}(G)$	set of edges e in G such that $f(e') = 1$ or $f(e'') = 1$ for directed edges e' and e'' corresponding to e in $\widetilde{G}$	108
$E^{f0}(G)$	set of edges e in G such that $f(e') = f(e'') = 0$ for directed edges e' and e'' corresponding to e in $\widetilde{G}$	108

$G_{f,k}$	spanning subgraph $(V, E^{f1}(G) \cup F_1 \cup F_2 \cup \cdots \cup F_k)$ of G, where $(F_1, F_2, \ldots, F_m)$ is a forest decomposition of $E^{f0}(G)$	108
$\lambda_s(G)$	s-proper edge-connectivity of graph G	117
$V_{a,b}$	set obtained from V by identifying $a, b \in V$ as a single element	130
$G/(u, v, \delta)$	graph obtained fron G by splitting edges $\{s, u\}$ and $\{s, v\}$ by weight δ	141
$\mathcal{C}(\alpha; G)$	set of all β-cuts in G satisfying $\alpha \leq \beta$	142
$\mathcal{C}_r(\alpha; G)$	set of all β-cuts X with $r \notin X$	142
$V_{(h,k)}$	$V_h \cup V_{h+1} \cup \cdots \cup V_k$ for an o-partition $(V_1, V_2, \ldots, V_r)$ and $1 \leq h \leq k \leq r$	146
G_s	graph obtained from G by eliminating s after isolating s in G	150
$\mathcal{C}(G)$	set of all minimum cuts in G	145
$V_{(h,k)}$	$V_h \cup V_{h+1} \cup \cdots \cup V_k$ for an ordered partition $(V_1, \ldots, V_r)$ and $1 \leq h \leq k \leq r$	146
$\delta(x, y)$	cycle distance between two nodes x and y in a cactus	162
$\Pi^3(\mathcal{C})$	set of all maximal circular MC partition of size 3 over cuts $\mathcal{C}$	168
$\mathcal{C}_{comp}(\pi)$	set of all minimum cuts in $\mathcal{C}(G)$ that are compatible with a partition π	174
$\mathcal{C}_{indv}(\pi)$	set of all minimum cuts in $\mathcal{C}(G)$ that are indivisible with a partition π	174
$\mathcal{X}(G)$	family of extreme vertex sets in an edge-weighted graph G	192
$\mathcal{X}_{B:A}$	$\{X \in \mathcal{X}(G) \mid X \subseteq V - A,\ X \cap B \neq \emptyset\}$ for disjoint subsets $A, B \subseteq V$	201
$\mathcal{T}_{B:A}$	tree representation for $\mathcal{X}_{B:A} \cup \{V\}$	201
$u^*(Y)$	one of the Y-minimizers	202
$\mathcal{X}_k(G)$	$\{X \in \mathcal{X}(G) \mid d(X; G) < k\}$	220
$c(G)$	number of components in G	238
$parity(v; G)$	0 if $d(v; G)$ is even, or 1 otherwise	243
$\mathcal{M}(G)$	set of all minimal minimum cuts in a graph G	247
$\Lambda_G(k)$	edge-connectivity augmentation function of a graph G	254
$[a, b]$	range from a to b	257
$\pi(R)$	size of a set R of ranges	257
$[a, b]\|^k$	upper k-truncation of a range $[a, b]$	257
$R\|^k$	upper k-truncation of a range set R	258
$[a, b]\|_k$	lower k-truncation of a range $[a, b]$	258
$R\|_k$	lower k-truncation of a range set R	258
$Ch(X)$	family of extreme vertex sets that are the children of X in the tree representation of the extreme vertex sets	260

$bot(r)$	bottom of a range r	265
$top(r)$	top of a range r	265
$\mathcal{E}_{k,\ell}$	family of all minimal deficient sets	289
$\hat{\kappa}^+(S, v; G)$	maximum number of internally vertex-disjoint directed paths from S to v	295
$\hat{\kappa}^-(S, v; G)$	maximum number of internally vertex-disjoint directed paths from v to S	295
T_f	time to evaluate the value of a set function f	307
$\mathcal{X}(f)$	family of all extreme subsets of a set function f	315
$P(f)$	polyhedron of a system (V, f)	322
$B(f)$	base polyhedron of a system (V, f)	322
$P_-(f)$	$P(f) \cap \Re^n_-$	322
$B_-(f)$	$B(f) \cap \Re^n_-$	322
$P_+(f)$	$P(f) \cap \Re^n_+$	322
$B_+(f)$	$B(f) \cap \Re^n_+$	322
$Ch(X)$	set of children of a set X in a laminar family	324
$pa(X)$	parent of a set X in a laminar family	324
$\mathcal{M}(\mathcal{X})$	set of all minimal subsets in a laminar family $\mathcal{X}$	326
$EP_-(f)$	set of all extreme points in $B_-(f)$	336
Π_n	set of all permutations of $(1, 2, \ldots, n)$	336
$L(f)$	set of all π-minimal vectors in $B_-(f)$ for each $\pi \in \Pi_n$	336
$\mathcal{M}(\mathcal{X}; X)$	set of all maximal subsets $Z \in \mathcal{X}$ with $Z \subseteq X$	337
$\mathcal{W}(f, g)$	family of all minimal deficient sets of (V, f, g)	342
$\mathcal{S}_v$	family of all v-solid sets	345
$\mathcal{S}(f)$	$\cup_{v \in V} \mathcal{S}_v$	345

Contents

1

Introduction

In Chapter 1, we introduce basic definitions and notions. We also outline some of the known algorithms devised for solving problems related to flows, cuts, and connectivities. These algorithms will be used as a basis for the discussion in subsequent chapters. The standard definitions and other topics in graph theory can be found in the book by R. Diestel [52] or other textbooks on graph theory (e.g., [10, 33]). For basic data structures for graphs, standard graph algorithms, and their complexity, see the book by R. E. Tarjan [298], for example.

1.1 Preliminaries of Graph Theory

Let $\Re$ (resp. $\Re_+$ and $\Re_-$) denote the set of reals (resp. nonnegative reals and nonpositive reals) and $\mathbf{Z}$ (resp. $\mathbf{Z}_+$ and $\mathbf{Z}_-$) denote the set of integers (resp. nonnegative integers and nonpositive integers). For a real $a \in \Re$, $\lceil a \rceil$ (resp. $\lfloor a \rfloor$) denotes the smallest integer not smaller than a (resp. the largest integer not larger than a). For two reals $a, b \in \Re$ with $a \leq b$, we denote by $[a, b]$ and (a, b) the closed interval and open intervals; i.e., the sets of reals c with $a \leq c \leq b$ and $a < c < b$, respectively.

A singleton set $\{x\}$ may be simply written as x, and "$\subset$" implies proper inclusion, whereas "$\subseteq$" means "$\subset$ or $=$". The union of a set A and a singleton set $\{x\}$ may be denoted by $A + x$.

Let V be a finite set. The cardinality of (i.e., the number of elements in) V is denoted $|V|$. Let 2^V denote the *power set* of V, i.e., the family of all subsets of V (hence $|2^V| = 2^{|V|}$). The set of all pairs of elements in a set V is denoted $\binom{V}{2}$ (hence $|\binom{V}{2}| = \binom{|V|}{2}$). We say that a subset $X \subseteq V$ *divides* another subset $Y \subseteq V$ if $X \cap Y \neq \emptyset \neq Y - X$. For two subsets $A, B \subset V$, we say that a subset $X \subseteq V$ *separates* A and B if $A \subseteq X \subseteq V - B$ or $B \subseteq X \subseteq V - A$. For two subsets $X, Y \subseteq V$, we say that X and Y *intersect* each other if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, and $Y - X \neq \emptyset$ hold, and we say that X and Y *cross* each other if, in addition, $V - (X \cup Y) \neq \emptyset$ holds. For a weight function $a : V \rightarrow \Re$, we denote $\sum_{v \in X} a(v)$ by $a(X)$ for all $X \subseteq V$. A set of subsets of V, $\{V_1, V_2, \ldots, V_k\}$ with

$V_i \subseteq V (i = 1, 2, \ldots, k)$, is a *partition* of V if $\bigcup_{i=1}^{k} V_i = V$ and $V_i \cap V_j = \emptyset$ holds for all $i \neq j$.

An *undirected graph* (or a *graph*) G and a *directed graph* (or a *digraph*) G are defined by a pair composed of a vertex set V and an edge set $E \subseteq V \times V$, depending on whether edges are undirected and directed, respectively, and are denoted by $G = (V, E)$. The vertex set and edge set of a graph G may be denoted by $V(G)$ and $E(G)$, respectively. We use the notation $n = |V|$ and $m = |E|$ throughout this book.

An undirected edge e with end vertices u and v is denoted by $\{u, v\}$. We say that e is *incident* with u and v, u and v are the end vertices of e, and u (resp. v) is *adjacent* to v (resp. u). A directed edge e with tail u and head v is denoted by (u, v), and the *head* (resp. *tail*) of e is denoted by $h(e)$ (resp. $t(e)$). In this case, we say that $e = (u, v)$ is incident from u to v. An edge with the same end vertex (u, v) is called a *loop*.

A (di)graph G is called *trivial* if $|V(G)| = 1$. A graph (resp. digraph) G is called *complete* if there is an edge $\{u, v\}$ (i.e., a pair of edges (u, v) and (v, u)) for every two vertices $u, v \in V(G)$. A (di)graph G is called *bipartite* if $V(G)$ can be partitioned into two subsets, V_1 and V_2, so that every edge has one end vertex in V_1 and the other in V_2.

Undirected edges with the same pair of end vertices (or directed edges with the same tail and head) are called *multiple edges*. A graph (resp. digraph) is called a *multigraph* (a *multiple digraph*) if it is allowed to have multiple edges; otherwise it is called *simple*. We sometimes treat a multigraph G as a simple graph with integer-weighted edges, where the weight of each edge $e = \{u, v\}$ represents the number of multiple edges with the same end vertices u and v. In such an edge-weighted representation, the number m of edges means the number of pairs of adjacent vertices in G.

The *degree* of a vertex v in G is the number of edges incident with v. If G is a digraph, the *indegree* (resp. *outdegree*) denotes the number of edges incident to (resp. from) v. The minimum degree (resp. maximum degree) of the vertices in G is denoted by $\delta(G)$ (resp. $\Delta(G)$). An undirected graph (resp. digraph) G is called *Eulerian* if the degree of each vertex is even (resp. the indegree is equal to the outdegree at every vertex).

A graph $G' = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$, which we denote by $G' \subseteq G$. G' is a *spanning subgraph* if $V' = V$. A subgraph $G' = (V', E')$ of $G = (V, E)$ is *induced* by V' if E' is given by $E' = \{e \in E \mid \text{both end vertices of } e \text{ belong to } V'\}$, and G' may be denoted by $G[V']$. Given an edge set F (not necessarily a subset of $E(G)$), we denote by $V[F]$ the set of end vertices of edges in F.

A sequence of vertices and edges in G, $P = (v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k)$, is called a *path* between v_1 and v_k (or from v_1 to v_k if G is a digraph) if $v_1, v_2, \ldots, v_k \in V$, $e_1, e_2, \ldots, e_{k-1} \in E$ and $e_i = \{v_i, v_{i+1}\}$ (or $e_i = (v_i, v_{i+1})$ if G is a digraph), $i = 1, 2, \ldots, k-1$. Such a path P is also denoted as a sequence of vertices $P = (v_1, v_2, \ldots, v_k)$ or a sequence of edges $P = (e_1, e_2, \ldots, e_{k-1})$ if

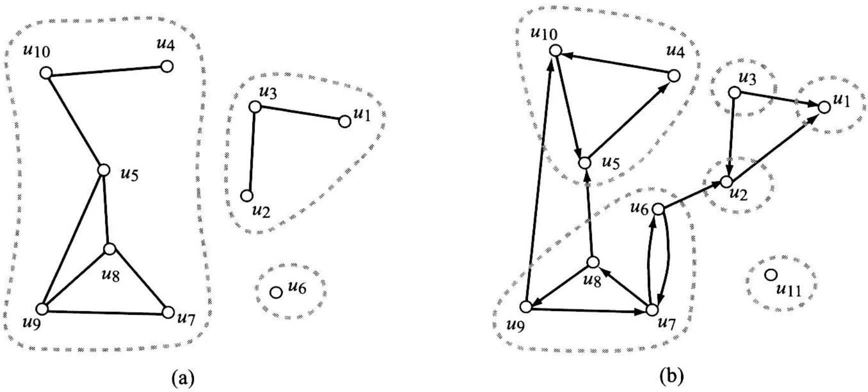

Figure 1.1. (a) A simple graph with three connected components; (b) a simple digraph with six strongly connected components, where each (strongly) connected component is enclosed by a gray dashed curve.

no confusion arises. For two vertices $u, v \in V$ in a graph (resp. digraph) G, a path between u and v (resp. a directed path from u to v) is called a *(u, v)-path*.

A graph (resp. digraph) G is called *connected* (resp. *strongly connected*) if G has a (u, v)-path for every pair of vertices u and v. A *connected component* (or a *component*) of a graph G is an inclusion-wise maximal vertex subset $X \subseteq V(G)$ such that every two vertices in X are connected by a path, where the induced subgraph $G[X]$ may also be called a (connected) component of G. A *strongly connected component* of a digraph G is an inclusion-wise maximal vertex subset $X \subseteq V(G)$ such that G has (u, v)- and (v, u)-paths for every two vertices u, $v \in X$, where the induced subgraph $G[X]$ may also be called a strongly connected component of G. Figure 1.1 illustrates examples of connected components of a graph and strongly connected components of a digraph.

An Eulerian connected graph has a sequence of edges by which we can visit all edges successively; we call such a sequence an *Eulerian trail*. Analogously an Eulerian strongly connected graph also admits an Eulerian trail, in which all directed edges are traversed along their directions. Figure 1.2(a) and (b) illustrate examples of Eulerian connected graph and strongly connected digraph, respectively.

1.1.1 Cut Functions of Weighted Graphs

When G is edge-weighted, the weight of an edge $e = \{u, v\}$ is denoted by $c_G(e)$ or $c_G(u, v)$, which are assumed to be nonnegative unless otherwise stated. Figure 1.3 shows a graph with integer edge weights, which can be viewed as a multigraph with multiplicity equal to the weight of each edge.

For two subsets $X, Y \subset V$ (not necessarily disjoint), $E(X, Y; G)$ denotes the set of edges e joining a vertex in X and a vertex in Y (i.e., $e = \{u, v\}$ satisfies $u \in X$ and $v \in Y$), and $d(X, Y; G)$ denotes $\sum_{e\in E(X,Y;G)} c_G(e)$. For a digraph $G = (V, E)$, we mean by $E(X, Y; G)$ the set of directed edges with a tail in X and