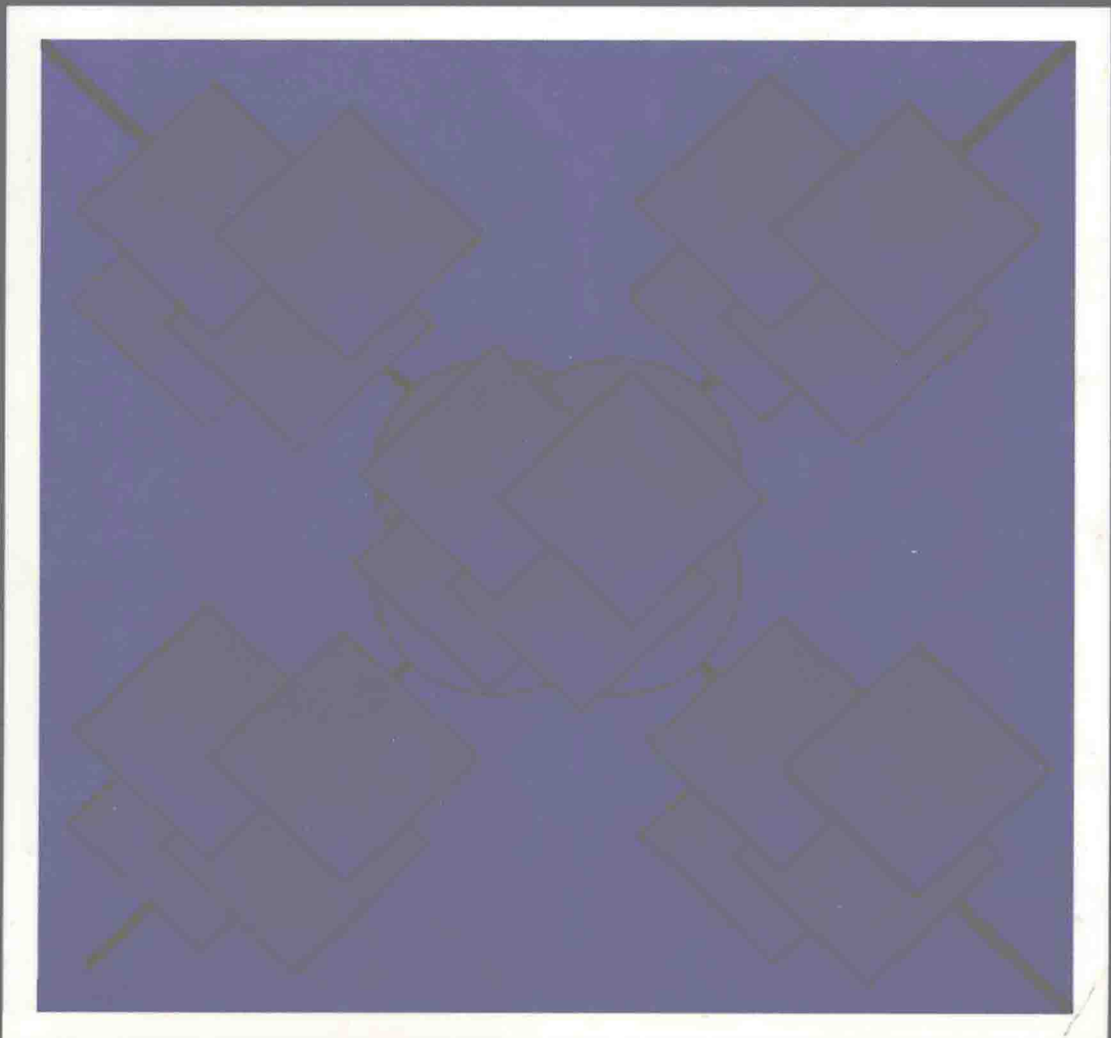
CALCULUS, CONCEPTS AND COMPUTERS PRELIMINARY VERSION

DUBINSKY SCHWINGENDORF



Calculus, Concepts and Computers Preliminary Version

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Ed Dubinsky
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Chapter 1

Getting Started

1.1 Preliminaries

This is a textbook for a Calculus course that uses computers. It uses computers for more than just calculating and working problems. The course uses computing machines and the things you do with them to help you *learn Calculus*. Keep that in mind as you use this book. You will learn how to solve many math problems by using your head to think and your machine to compute. You will also learn to think in mathematical ways. That means dealing with a problem situation first by understanding what is going on and then by using the simplest method you can think of to handle the situation. Sometimes you will use a method you used before. Sometimes you will have to modify an old one or make one up that is entirely new to you.

One approach of this book is to use computers to stimulate your thinking about strategies and methods for solving problems and to help you develop skills to implement them. Another is to confront you with the kind of questions that will force you to think about problems and how to solve them.

1.1.1 What you need to know about computers

You don't have to know anything about computers in order to take this course. We start at the very beginning. The goal of this first chapter is to give you a chance to get to know the computer systems you will be using. We also want you to have an introduction to many of the problems, methods and mathematical ideas that you will be dealing with throughout the course.

We will take you through these computer systems very slowly and carefully in this chapter. This will give you a chance to really get to know them and to develop an initial facility in using them.