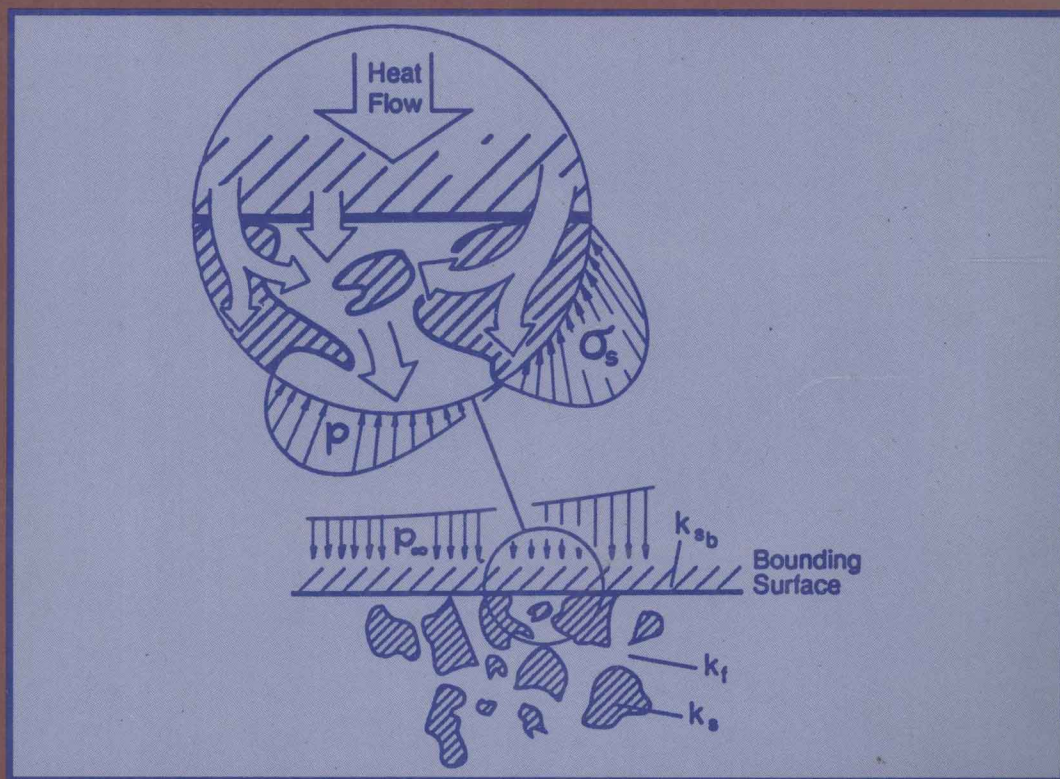# COMPUTATIONAL MECHANICS OF POROUS MATERIALS AND THEIR THERMAL DECOMPOSITION

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COMPUTATIONAL MECHANICS OF POROUS MATERIALS AND THEIR THERMAL DECOMPOSITION

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FOREWORD

Numerical models of porous media are continuum analogs of two mathematical formulations: Biot's theory and the theory of mixtures.

The concepts and thinking underlying the mathematical model and the derivation of general, coupled chemo-thermoelastic equations based on Biot's theory are presented by Weiler. As one studies this formulation and reads the applications, the number of physical constants becomes mind-boggling, despite the fact that Biot's theory is designed to simplify the two-phase phenomena. It becomes obvious that theory and numerics is worthless without adequate experimentation. Fortunately, several papers here treat experimental aspects of the problem for ablative phenolic materials.

An alternative approach, the theory of mixtures, is applied to two problems in biomechanics. Gu, Lai and Mow obtain numerically the concentration of ions, fluid and ion velocities and the strain field inside due to mechanical and osmotic pressure outside of a triphasic mechano-electro-chemical layer. R. L. Spilker and E. S. de Almeida Neto develop a finite element formulation for the governing equations for a biphasic incompressible hyperelastic, finite deformable solid and incompressible, inviscid fluid. This more general theory offers fresh insight to the engineered and geological materials community, but it remains to be seen if it simplifies matters. If these theories tend to overly smear out local detail and your interest is in the details of energy transfer, then read Kaviany. He provides heat transfer (all modes) and fluid flow equations for periodic structures, particularly for packed beds of spherical particles.

In practical cases, solution of these numerical systems requires extensive computing resources, hence it is little surprise that thermo-poroelastic applications presented here are limited to one-dimensional problems. Nonetheless, more general problems may be within reach. The path is pioneered by the longer established soil mechanicians. For instance, Mish, Herrmann and Mura-leetharan treat multi-dimensional soil dynamics problems using several forms of Biot's theory and discuss their time integration. Their interest is in development of efficient finite element codes capable of accurate calculation of response, moreover free of spurious high frequency oscillations.

The fundamental model of thermal decomposition of phenolic resin composites couples a single conservation of energy equation with mass continuity and conservation of gas momemtum equations, the latter featuring an equation of Darcy or Darcy-Forcheimer type. Deformation of the material system is neglected. Keyhani and Krishnan use such a model to study solution sensitivity to highly variable (and uncertain) thermophysical properties and weakly-and fully-coupled equations as reflected in results for pore pressure and thermal advection.

Adding material expansion to the above fundamental model, Henderson, Florio and Miriyale couple separate conservation of energy equations for solid and gas, mass continuity and conservation of gas momemtum featuring a Darcy-Forcheimer type equation. Expansion of the material is accounted for parametrically and as a function of time; this simplification eliminates solving or the solid momemtum, yet affects the solution interactively in a stepwise manner.

The most complete model is one featuring chemo-thermo-poroelastic behavior. Employing such a model, Wu and Katsube develop, step-by-step, a micromechanical formulation of the problem. Their inclusion of elastic deformation is focused upon pore volume change due to decomposition and shrinkage. McManus, using both pressure dependent and independent Arrhenius forms to model pyrolysis, and Biot's theory for deformation, provides boh analytical and numerical results with suggestions for improvement of the latter. Sullivan employs a finite element code to investigate the physics of decomposing phenolic composites with respect to various parameters. Of particular interest is the effect of elastic strain.

As mentioned above, the importance of experimental observation to theory and modeling is essential. For this reason, experimentalists were invited to contribute data on phenolic composites. Brown and Clemons offer substantial documentation on the process dynamics of carbon-carbon from carbon-phenolic. Stokes characterizes the permeability of carbon phenolic in two temperature regimes by studying the material microstructure.

In closing, the editors apologize for the proliferation of symbols and the confusion they may cause. On the other hand, we thank the authors for their timely contributions. We think you will find this volume worthwhile.

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CONTENTS

FULLY COUPLED THERMO-PORO-ELASTO
GOVERNING EQUATIONS

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ABSTRACT

The objective of the present paper is to outline the development of a set of *thermo-poro-elasto* governing equations which represents the physical theory upon which new advanced rocket nozzle analysis computer programs will be based. The three relevant areas of interest are (a) nonlinear thermo-poro-structural analysis, (b) nonlinear pyrolysis gas flow, and (c) nonlinear heat transfer, all of which pertain to rocket nozzle ablative materials such as carbon-phenolics.

INTRODUCTION

Due to the nature of this paper, it is presented in 1-column, full width format, mainly because of the large number of equations required to outline the development. It is beyond the scope of the present paper to present an in depth review of all of the theory behind the relevant equations which govern the phenomenon of how rocket nozzle ablative materials behave. The present paper represents a condensed version of a previous paper (Reference [1]) wherein complete derivations are presented. Some of the material contained in [1] was drawn from references [3] through [7], which formed the basis of the original paper [2]. However, much has been derived as an extension to these references, incorporating appropriate formulations where deemed necessary. References [8-32] list but a few of the vast extent of the literature relevant to the "pore presure" phenomenon. The reader is referred to these sources [1-32] for more detailed information complementary to the current report.

The governing equations will be developed in sequence for (a) the concepts of a porous medium, (b) the nonlinear thermo-poro-structural response, (c) the nonlinear pyrolysis gas generation and flow, and (d) the nonlinear heat transfer within the material. This will be followed by a section discussing the coupling which exists between these three disciplines. Finally, the governing equations (which were cast into finite element form) will be shown and appropriate solution algorithms will be postulated.

The author will apologize up front for being *pedantic*, but felt that many of the detailed explanations were necessary to make absolutely clear what assumptions were involved, and exactly what was was being derived. All too often, many of the articles in the literature do not clearly state implied assumptions, and also do not make it clearly known what is being derived. Many times a final equation is stated without any reference to where it came from. These practices have left the reading audience bewildered and confused. The author hopes that this is not the case with the present paper.

Before proceeding, a few words about the notation used in this paper. Most of the time, matrix and vector notation will be used instead of indical notation (who can ever keep track of all of those subscripts?). To help distinguish between vectors and matrices, the convention of *lowercase* characters representing vectors and *UPPERCASE* characters representing matrices will be used, with the possibility of a few exceptions. In addition, both lowercase and uppercase **boldface** characters represent vectors and matrices in the same manner as if the appropriate braces $\{*\}$ for column vectors, floors $\lfloor * \rfloor$ for row vectors, brackets $[*]$ for matrices and $\lceil * \rceil$ for diagonal matrices were used. Also, a superscript "T" represents the *transpose* operation on both vectors and matrices.

Solid Skeleton, Pores & Porosity

The medium of interest is that of a porous material, which consists of a solid skeleton (sometimes referred to as the solid phase) and voids or porosity which contains a fully saturated gas (or fluid). The subject of a partially saturated gas (or fluid) could equally be treated, but the problem of interest is for a fully saturated one. The gas is assumed to distribute pressure evenly to all surfaces of the solid skeleton. In addition, only those pores which are interconnected are included in the *pore volume*, and are considered separately from the solid skeleton. Those pores which are not interconnected are considered as part of the solid skeleton. This assumption is usually referred to as *effective porosity* within the literature.

The ensuing governing equations will be developed for a representative control volume (RCV) which is assumed large enough such that the exact nature and arrangement of the individual pores is unimportant. However, the RCV is still small enough so it may be treated as an infinitesimal volume. The porosity (or pores) need not be constant throughout the body, but are assumed to vary slow enough so they may be considered constant within the RCV.

Now, the total volume V in the RCV is the sum of the pore volume V_p and the solid volume V_s, i.e.,

$$ V = V_s + V_p, \qquad \varphi = \frac{V_p}{V}, \qquad (1 - \varphi) = \frac{V_s}{V} \qquad (1a,b,c) $$

where the *volume* porosity φ has been defined as the ratio of the pore volume to the total control volume. There are two types of porosity used in the following derivations. There is a *volume porosity* φ defined above and an *areal porosity* ω_i which will be explained next.

In the typical RCV, the interconnected void spaces (pores) form the *pore volume*. However, the tortuous paths of these interconnected pores are far from being uniform, and possess maximum and minimum cross sectional areas. If this RCV was cut by a series of parallel planes, then the cut surfaces would produce different *cut pore areas* on each of the cut planes. The ratio of the

cut pore area to the total cut cross-sectional area is defined as the *areal porosity* ω_i and is given by (i = coordinate direction)

$$\omega_i = \frac{\text{cut pore area on the } i^{th} \text{ face}}{\text{whole area on the } i^{th} \text{ face}} \qquad (2)$$

For the present, they are considered different for the different principal directions of the material. In the cartesian (x,y,z) coordinate system, the areal porosities are denoted $\omega_x, \omega_y,$ and ω_z. To further illustrate, if the areal porosities of these cut planes were plotted versus the distance across the RCV, they would appear random in nature as shown in Figure 1.
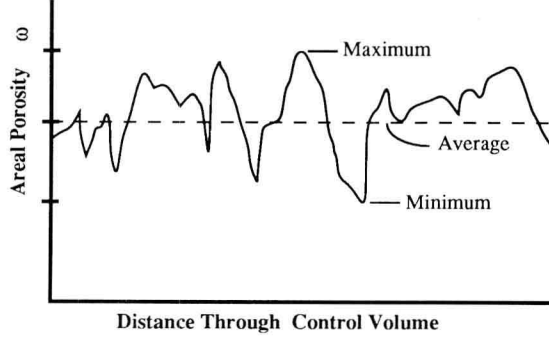

Figure 1 - Variation of Areal Porosity through the RCV

One may notice that *maximum* and *minimum* values occur, even within this representative control volume. For the purposes of the following derivation, two types of areal porosity *extremes* will be defined:

> **force areal porosity** - represents all of the areal porosities which lie *above* some upper threshold (e.g., the top 5%) and are designated by ω_i.

> **flow areal porosity** - represents all of the areal porosities which lie *below* some lower threshold (e.g., the bottom 5%) and are designated by v_i.

In reality, if one were to integrate the cut pore area (or areal porosity) distribution shown in Figure 1 along the axis perpendicular to the cuts, the resulting volume (or value) would be the pore volume (or volume porosity). Consider

$$\int A_x^p(x)\,dx = V_x^p = \widetilde{\omega}_x V$$

$$\int A_y^p(y)\,dy = V_y^p = \widetilde{\omega}_y V$$

$$\int A_z^p(z)\,dz = V_z^p = \widetilde{\omega}_z V$$

where A_i^p represents the cut pore area for the *i-th* direction cut face (x, y or z). Since the volume of the pores is single valued, then $V_x^p = V_y^p = V_z^p$, causing the *average* areal porosities $\widetilde{\omega}_x = \widetilde{\omega}_y = \widetilde{\omega}_z = \varphi$. This fact would be true regardless of the direction of the cuts. This is why the literature *always* treats the areal porosity equal to the volume porosity. What really is meant is that the *average* areal porosity within the RCV is equal to the volume porosity φ.

However, as will be seen later in the derivation, the *maximum* areal porosity is associated with the force balance equations (stresses) since the minimum solid skeleton area corresponds to the maximum pore area, which therefore greatly influences the local state of stress in the solid skeleton. On the other hand, the *minimum* areal porosity is associated with the flow of the pyrolysis gases through the porous material, and therefore it is the minimum pore area which greatly influences the state of this flow. This is the reasons given for the name definitions presented above. More about these concepts will be discussed later.

Nonlinear Thermo-Poro-Structural Equations

Since matrix notation will be used extensively in the following derivation of the governing thermo-poro-structural equations, the following vector definitions for stress σ and total strain ϵ will be made, with similar definitions applying for mechanical strain ϵ^m, pore-pressure strain ϵ^{pr}, thermal strain ϵ^{th}, and creep strain ϵ^c. In addition, the vector form of the Kronecker delta contraction function δ, the areal porosity vector ω and matrix $\lceil \mathbf{W} \rfloor$ (explained below) and the identity diagonal matrix $\lceil \mathbf{I} \rfloor$ will also be made. They are presented here (and throughout this paper) in the cartesian (x,y,z) coordinate system. They will now be defined for convenience in row vector notation by

$$\{\sigma\}^T = \lfloor \sigma_x \;\; \sigma_y \;\; \sigma_z \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{zx} \rfloor$$

$$\{\epsilon\}^T = \lfloor \epsilon_x \;\; \epsilon_y \;\; \epsilon_z \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx} \rfloor$$

$$\delta^T = \lfloor 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \rfloor$$

$$\omega^T = \lfloor \omega_x \;\; \omega_y \;\; \omega_z \;\; 0 \;\; 0 \;\; 0 \rfloor \qquad\qquad (3a)$$

$$\lceil \mathbf{W} \rfloor = \lceil \omega_x \;\; \omega_y \;\; \omega_z \;\; \omega_x \;\; \omega_y \;\; \omega_z \rfloor$$

$$\lceil \mathbf{I} \rfloor = \lceil 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rfloor$$

The last two definitions are for a *diagonal matrix* $\mathbf{W}$ and $\mathbf{I}$ where the notation $\lceil ... \rfloor$ was used. Notice that the δ, ω and $\lceil \mathbf{W} \rfloor$ definitions are related by

$$\omega = \lceil \mathbf{W} \rfloor \delta \qquad\qquad \omega^T = \delta^T \lceil \mathbf{W} \rfloor \qquad\qquad (3b)$$

Mechanical, Equivalent and Total Stress Definitions

Figure 2 depicts a representative control volume (RCV) of a porous material loaded by both mechanical stresses σ_{ij}^m and pore pressure p. There are two distinct boundaries associated with this RCV. The *outer* boundary B_o and the *interior* boundary B_i. The outer boundary consists of two parts;

The outer solid boundary B_{so} which represents that portion of the cut faces where the solid skeleton material exists. The area of this outer boundary is the *solid area* A_{so}.

The outer pore boundary B_{po} which represents that portion of the cut faces where the (interconnected) pores exists. The area of this outer boundary is the *pore area* A_{po}.

The interior boundary represents all of the boundaries between the (interconnected) pores and the solid skeleton. Those pores which are not interconnected are considered part of the solid skeleton. These boundary and area definitions will now be used to define the loading which exists on the RCV. The mechanical stresses σ^m are assumed to act only on the solid skeleton, and therefore, the mechanical surface tractions act only on A_{so}, i.e., the B_{so} portion of B_o. The pore pressure p is assumed to act on A_{po}, i.e., on the B_{po} portion of the outer boundary B_o, and also on all of the interior boundary B_i.
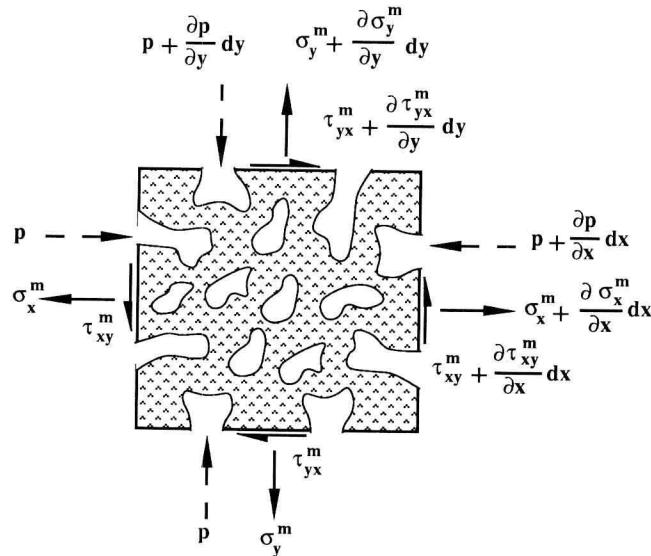
Figure 2 - Representative Control Volume & Loadings

The RCV shown in Figure 2 may be considered as a 2-dimensional (x-y) model, with the x-axis horizontal (to the right), the y-axis vertical (upward) and the z-axis coming out of the paper. It is understood that the incremental lengths of this RCV are dx, dy and dz. The 2-dimensional x-force and z-moment equilibrium equations will be developed for this model. The remaining 2-dimensional

equilibrium equations may easily be derived by the same procedure, but will not be developed here. The objective is to determine the 3-dimensional equilibrium equations, whose development will also not be shown, since they are similar to the 2-dimensional case. Prior to the derivation, the following quantities are defined.

$dA_s^x = (1-\omega_x)\,dy\,dz$ = solid area on the cut x-face
$dA_s^y = (1-\omega_y)\,dx\,dz$ = solid area on the cut y-face
$dA_p^x = \omega_x\,dy\,dz$ = pore area on the cut x-face
$dA_p^y = \omega_y\,dx\,dz$ = pore area on the cut y-face
$dV_s = (1-\phi)\,dV$ = solid volume within RCV
$dV_p = \phi\,dV$ = pore volume within RCV

where $\omega_i = A_p^i/A_i$ are the *areal porosities* and ϕ is the volume porosity. Now, requiring the sum of the forces in the x-direction to vanish, remembering pressure is positive in compression, and dividing through by the incremental volume $dx\,dy\,dz$ and collecting similar terms leads to

$$(1-\omega_x)\frac{\partial}{\partial x}\overset{m}{\sigma}_x + (1-\omega_y)\frac{\partial}{\partial y}\overset{m}{\tau}_{xy} - \omega_x\frac{\partial p}{\partial x} + (1-\phi)X_s + \phi X_p = 0 \tag{4a}$$

Now, requiring the sum of the moments about the z-axis to vanish, dividing through by the incremental volume $dx\,dy\,dz$, collecting similar terms and recognizing that as $dx \to 0$ and $dy \to 0$, terms multiplied by dx and dy vanish, leads to

$$(1-\omega_x)\,\overset{m}{\tau}_{yx} = (1-\omega_y)\,\overset{m}{\tau}_{xy} \tag{4b}$$

If equivalent stresses σ_x, τ_{xy}, τ_{yx} and an equivalent body force X are defined by

$$\sigma_x = (1-\omega_x)\overset{m}{\sigma}_x - \omega_x p \qquad X = (1-\phi)X_s + \phi X_p$$
$$\tau_{yx} = (1-\omega_x)\overset{m}{\tau}_{yx} \qquad \tau_{xy} = (1-\omega_y)\overset{m}{\tau}_{xy} \tag{5}$$

then the force equilibrium equation (4a) and the moment equilibrium equation (4b) will become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$
$$\tau_{xy} = \tau_{yx} \tag{6}$$

These are the equilibrium equations found in textbooks on solid elasticity theory. The other 2-dimensional force and moment equilibrium equations may be written down by inspection. The 3-dimensional equations of equilibrium may be derived from these 2-dimensional equations. Before stating these equations the 3-dimensional *equivalent stress* definitions similar to those in equations (5) will be made. They are

$$\sigma_x = (1-\omega_x)\overset{m}{\sigma}_x - \omega_x p \qquad X = (1-\phi)X_s + \phi X_p$$
$$\sigma_y = (1-\omega_y)\overset{m}{\sigma}_y - \omega_y p \qquad Y = (1-\phi)Y_s + \phi Y_p$$
$$\sigma_z = (1-\omega_z)\overset{m}{\sigma}_z - \omega_z p \qquad Z = (1-\phi)Z_s + \phi Z_p$$
$$\tau_{xy} = (1-\omega_x)\overset{m}{\tau}_{xy} \qquad \tau_{yx} = (1-\omega_y)\overset{m}{\tau}_{yx}$$
$$\tau_{yz} = (1-\omega_y)\overset{m}{\tau}_{yz} \qquad \tau_{zy} = (1-\omega_z)\overset{m}{\tau}_{zy}$$
$$\tau_{zx} = (1-\omega_z)\overset{m}{\tau}_{zx} \qquad \tau_{xz} = (1-\omega_x)\overset{m}{\tau}_{xz} \tag{7}$$

Using the above definitions for equivalent stress, the 3-dimensional equilibrium equations are just extensions of the 2-dimensional equations shown in equations (6). They will simply be stated with any development left to the reader. The force equilibrium equations are given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$
$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \tag{8a}$$

Similarly, the moment equilibrium equations are given by

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$
(8b)

Therefore, by defining *equivalent stresses* as shown in equations (7), it is *these* stresses which obey the equilibrium equations found in the theory of solid elasticity.

It is important to note that when the areal porosities ω_i are *assumed* different in the principal material directions, then the actual mechanical shear stresses τ_{ij}^m acting on the solid skeleton are different even though the average shear stresses τ_{ij} are the same, as shown by equations (7) and (8b). Of course when the areal porosities are all the same, say $\omega = \varphi$ the volume porosity as explained above, then the actual mechanical shear stresses τ_{ij}^m are the same, differing only in magnitude from the average shear stresses τ_{ij} by the factor $(1 - \varphi)$.

Another interpretation may be made for the equivalent stress definitions given in equations (7). Notice that these equivalent stresses are an *area weighted average* (through use of the areal porosities ω_i) of the solid skeleton stresses σ_{ij}^m and the pore pressure p. They could therefore just as easily be called the *total average stresses* σ acting on the RCV rather than the equivalent stresses. The two will be treated equivalently in what follows. The equivalent stress definitions given in equations (7) may be restated in vector notation as (where $\lceil \mathbf{I} \rfloor$, $\lceil \mathbf{W} \rfloor$ and ω are defined in equations (3))

$$\sigma = \lceil \mathbf{I} - \mathbf{W} \rfloor \sigma^m - \omega p \qquad \text{pore anisotropy}$$
$$\sigma = (1 - \varphi) \sigma^m - \varphi \delta p \qquad \text{pore isotropy}$$
(9)

Throughout the remainder of this section, dual sets of equations will be presented. The first set will include the possibility that the *areal porosities* **may** be different in the principal directions of the material (where at least the *local* maximum is in fact different). Since this section is developing the thermo-poro-structural equations, the *force areal porosity* ω_i will be used. These equations will be labeled as **pore anisotropy**. The second set of equations will include the usual assumption that the areal porosities are the same in all directions, and therefore be labeled **pore isotropy**.

Before proceeding with the development of the constitutive equations for these stress variables, an alternate definition of stress will be made. Towards this end, consider Figure 3 which highlights the top face of the RCV. The stresses in this figure are by definition the forces acting on the face of the RCV divided by the *total* area of the face. In the first RCV in Figure 3, the total average stress σ acts over the top face and the pore pressure p acts normal to all interior surfaces of the solid skeleton, i.e., those interior surfaces which form part of the interconnected pores. In the second RCV, the *average mechanical stress* σ^a is assumed to act only on the solid skeleton but is averaged over the whole outer surface area $A_o = A_{so} + A_{po}$. It is also important to note that the open portion of the pores on this face (defined as A_{po} above) have no applied loads. In the third RCV, the pore pressure p is assumed to act on all interior pore surfaces and on the whole outer surface area A_o.

| σ | σ^a | p |

Total Stress acts on the whole area, presure acts on the pores and faces

Mechanical Stress acts only on the solid skeleton, but *averaged* over whole area

Pressure Stress applied everywhere but only normal to the boundary
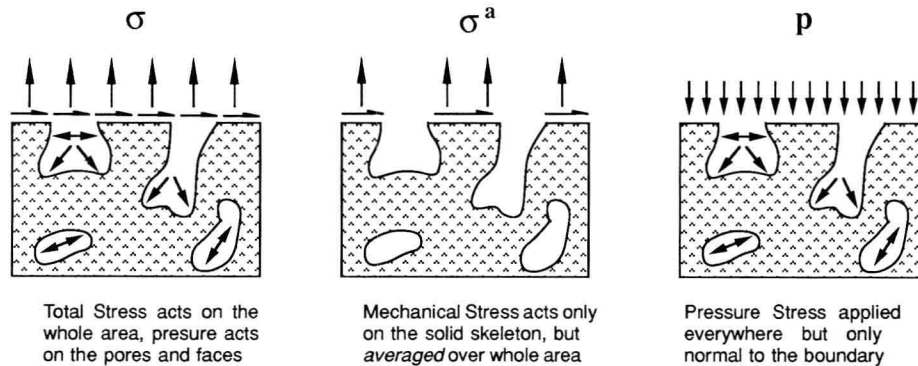
Figure 3 - Decomposition of Applied Stresses & Pressure

Therefore, when viewing Figure 3, it becomes apparent that the total loading (first RCV) may be decomposed into a pure pressure loading (third RCV) and the remaining *average* mechanical stress loading (second RCV). This is simply a force decomposition since all three stress measures use the same total outer surface area A_o in their definition. Consequently

$$\sigma = \sigma^a - \delta p \qquad \text{pore anisotropy}$$
$$\sigma = \sigma^a - \delta p \qquad \text{pore isotropy}$$
(10)

where δ is the vector form of the Kronecker delta function defined in equations (3). Remember, the sign convention is that the stresses are positive in tension whereas the pressure is positive in compression, thereby causing the minus sign for the pressure

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term in (10). The pressure is a direct stress and therefore is added only to the direct stress $(\sigma_x, \sigma_y, \sigma_z)$ components, hence the use of the Kronecker delta function in this equation.

This alternate definition of stress acting on the RCV is one for convenience in the development that follows. Remember, σ^a is the *average mechanical stress* acting on the solid skeleton and is defined by the *whole* outer surface area A_o. In contrast, the actual mechanical stress σ^m defined in equation (9) acts only on the outer solid skeleton area A_{so} (see above). With these definitions, the *actual* mechanical stress σ^m may be related to the *average* mechanical stress σ^a and the pore pressure p by the following. Equating the total average stress definitions given by equations (9) and (10) leads to

$$\sigma^m = \lceil \mathbf{I} - \mathbf{W} \rfloor^{-1} [\sigma^a - (\delta - \omega)p]$$
$$\sigma^a = \lceil \mathbf{I} - \mathbf{W} \rfloor \sigma^m + (\delta - \omega)p \qquad \text{pore anisotropy} \qquad (11a)$$

If the areal porosities ω_i were all equal, say to ω which is equal to the volume porosity φ as explained above, then these equations would reduce to

$$\sigma^m = \frac{1}{(1-\varphi)}\sigma^a - \delta p$$
$$\sigma^a = (1-\varphi)[\sigma^m + \delta p] \qquad \text{pore isotropy} \qquad (11b)$$

The last interesting point to be made concerning stresses is the following: it is the total average stress σ that is used in the force equilibrium equations where it is usually the mechanical stress σ^m (or σ^a) that relates stress to strain (constitutive law).

Mechanical Stress and Strain Relationships

The relationships between the actual mechanical stress σ^m, the average mechanical stress σ^a the pore pressure p, and the strains within the solid skeleton will be derived. In order to accomplish these definitions, *strain energy relationships* will be used.

First, the actual mechanical strain ϵ^m and the average mechanical strain ϵ^a are defined as those strains resulting from the application of σ^m and σ^a respectively, in a *dry unsaturated porous material*. Since the fluid medium in the present derivation is pyrolysis gas and not a liquid (like water or oil), then the two states of a *dry unsaturated porous material* and a *wet saturated porous material at ambient (zero) pressure* are one in the same. This of course assumes that the presence of the pyrolysis gases do not alter (in any appreciable way) the behavior of the solid material.

The definition of *strain energy* within a representative control volume (RCV) is defined in vector notation by

$$U = \int_V \left[\int dU \right] dV = \int_V \left[\int_0^{\epsilon} (\sigma^a)^T d\epsilon^a \right] dV$$

Assuming stress is linearly related to strain (e.g., a linear stress-strain matrix $\mathbf{C}_m$), then the total strain energy within the RCV is given by

$$U = \int_V \left[\tfrac{1}{2}(\sigma^a)^T \epsilon^a \right] dV = \tfrac{1}{2}(\sigma^a)^T \epsilon^a V \qquad (12)$$

In the absence of pressure, the average mechanical stress σ^a is related to the actual mechanical stress σ^m by (see equations (11)).

$$\sigma^a = \lceil \mathbf{I} - \mathbf{W} \rfloor \sigma^m \qquad \text{or} \qquad \sigma^m = \lceil \mathbf{I} - \mathbf{W} \rfloor^{-1} \sigma^a \qquad \text{pore anisotropy}$$
$$\sigma^a = (1-\varphi)\sigma^m \qquad \text{or} \qquad \sigma^m = \frac{1}{(1-\varphi)}\sigma^a \qquad \text{pore isotropy} \qquad (13)$$

Substituting these stress definitions into equation (12), dropping the 1/2 factor, using equation (1c) to relate the solid skeleton volume V_s to the total volume V and noting that *by definition*, the actual mechanical strain ϵ^m within the solid skeleton due to the actual mechanical stress σ^m produces the strain energy $U = (\sigma^m)^T \epsilon^m V_s$, leads to the equations which state that the average mechanical strain ϵ^a is related to the actual mechanical strain ϵ^m by the following.

$$\epsilon^m = \frac{1}{(1-\varphi)}\lceil \mathbf{I} - \mathbf{W} \rfloor \epsilon^a \qquad \text{pore anisotropy}$$
$$\epsilon^m = \epsilon^a \qquad \text{pore isotropy} \qquad (14)$$

This equality for the *pore isotropy* case is also confirmed by the fact that both the average and actual strains are related to the *same* displacements on the outer boundary of the RCV, and these strain-displacement relationships are simply kinematical, regardless of the makeup of the RCV. The weight factor $\lceil \mathbf{I} - \mathbf{W} \rfloor/(1-\varphi)$ for the *pore anisotropy* case simply relates the relationship necessary for the strain energy equality.

Stress-Strain Relationships

The average mechanical strain ϵ^a is related to the average mechanical stress σ^a (no pressure present) within the RCV by the *compliance* or *flexibility* matrix $\mathbf{D}_m$ of the porous material. A subscript "m" will denote properties of the porous material. Therefore, this relationship is given in both incremental and total form by

$$d\epsilon^a = \mathbf{D}_m\, d\sigma^a \tag{15a}$$

$$\epsilon^a = \tilde{\mathbf{D}}_m\, \sigma^a \tag{15b}$$

The inverse of this strain-stress relationship is the stress-strain relationship, which relates average mechanical stress σ^a to average mechanical strain ϵ^a by the *stiffness* matrix $\mathbf{C}_m$, i.e.,

$$d\sigma^a = \mathbf{C}_m\, d\epsilon^a \tag{16a}$$

$$\sigma^a = \tilde{\mathbf{C}}_m\, \epsilon^a \tag{16b}$$

where $\mathbf{C}_m = \mathbf{D}_m^{-1}$. The form of the incremental matrices $\mathbf{D}_m$ or $\mathbf{C}_m$ may be linear or nonlinear, with dependence upon mechanical stress, pore pressure, temperature, etc., whereas the total matrices $\tilde{\mathbf{D}}_m$ or $\tilde{\mathbf{C}}_m$ is considered linear, with only a dependence upon temperature, i.e.,

$$\mathbf{D}_m = \mathbf{D}_m(\sigma^m, p, T) \qquad \mathbf{C}_m = \mathbf{C}_m(\sigma^m, p, T)$$

$$\tilde{\mathbf{D}}_m = \tilde{\mathbf{D}}_m(T) \qquad \tilde{\mathbf{C}}_m = \tilde{\mathbf{C}}_m(T)$$

The form of these matrices is presented in Appendix A, where the coefficients are expressed in terms of the usual engineering constants. If the incremental matrices $\mathbf{D}_m$ or $\mathbf{C}_m$ were linear, without dependence upon mechanical stress σ^m or pressure p, then the incremental form of these relationships could be integrated yielding the total form shown in equations (15b) and (16b). Therefore, the following notation will be used to describe the differences between incremental and total forms of these relationships;

   $\mathbf{M}$ - represents *incremental* properties where the coefficient matrices/vectors are *nonlinear*

   $\tilde{\mathbf{M}}$ - represents *total* properties where the coefficient matrices/vectors are considered *linear*

The relationships between the *actual* mechanical strains ϵ^m and *actual* mechanical stresses σ^m within the *solid skeleton* are the same as those stated above for the *porous material*. A subscript "s" will denote properties of the solid material. Therefore, the solid skeleton material counterpart to equations (15) and (16) are

$$d\epsilon^m = \mathbf{D}_s\, d\sigma^m \qquad \epsilon^m = \tilde{\mathbf{D}}_s\, \sigma^m \tag{17a,b}$$

$$d\sigma^m = \mathbf{C}_s\, d\epsilon^m \qquad \sigma^m = \tilde{\mathbf{C}}_s\, \epsilon^m \tag{18a,b}$$

The porous material stiffness matrix $\tilde{\mathbf{C}}_m$ may be related to the solid material stiffness matrix $\tilde{\mathbf{C}}_s$ by substituting for the average mechanical stress σ^a from equation (13) and the average mechanical strain ϵ^a from equation (14) into the stress-strain relationship expressed in equation (16b), and using the stress-strain relationship in equation (18b) to yield

$$\lceil \mathbf{I} - \mathbf{W} \rfloor \sigma^m = \tilde{\mathbf{C}}_m (1 - \varphi) \lceil \mathbf{I} - \mathbf{W} \rfloor^{-1} \epsilon^m = \lceil \mathbf{I} - \mathbf{W} \rfloor \tilde{\mathbf{C}}_s\, \epsilon^m \qquad \text{pore anisotropy}$$

$$(1 - \varphi)\sigma^m = \qquad \tilde{\mathbf{C}}_m\, \epsilon^m \qquad = (1 - \varphi)\tilde{\mathbf{C}}_s\, \epsilon^m \qquad \text{pore isotropy}$$

Therefore, these equations yield the following relationships between $\tilde{\mathbf{C}}_m$ and $\tilde{\mathbf{C}}_s$, namely

$$\tilde{\mathbf{C}}_m = \frac{1}{(1 - \varphi)} \lceil \mathbf{I} - \mathbf{W} \rfloor \tilde{\mathbf{C}}_s \lceil \mathbf{I} - \mathbf{W} \rfloor \qquad \text{pore anisotropy}$$

$$\tilde{\mathbf{C}}_m = \qquad (1 - \varphi)\tilde{\mathbf{C}}_s \qquad \text{pore isotropy} \tag{19a}$$

or the inverse relationships

$$\tilde{\mathbf{C}}_s = (1 - \varphi)\lceil \mathbf{I} - \mathbf{W} \rfloor^{-1} \tilde{\mathbf{C}}_m \lceil \mathbf{I} - \mathbf{W} \rfloor^{-1} \qquad \text{pore anisotropy}$$

$$\tilde{\mathbf{C}}_s = \qquad \frac{1}{(1 - \varphi)} \tilde{\mathbf{C}}_m \qquad \text{pore isotropy} \tag{19b}$$

Similarly, the relationship between the flexibility matrices $\tilde{\mathbf{D}}_m$ and $\tilde{\mathbf{D}}_s$ may be found by using equations (13), (14), (15b) and (16b) yielding

$$\widetilde{\mathbf{D}}_m = (1-\varphi)\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1} \widetilde{\mathbf{D}}_s \lceil \mathbf{I} - \mathbf{W}\rfloor^{-1} \qquad \text{pore anisotropy}$$

$$\widetilde{\mathbf{D}}_m = \frac{1}{(1-\varphi)} \widetilde{\mathbf{D}}_s \qquad\qquad \text{pore isotropy} \qquad (20a)$$

or the inverse relationships

$$\widetilde{\mathbf{D}}_s = \frac{1}{(1-\varphi)}\lceil \mathbf{I} - \mathbf{W}\rfloor \widetilde{\mathbf{D}}_m \lceil \mathbf{I} - \mathbf{W}\rfloor \qquad \text{pore anisotropy}$$

$$\widetilde{\mathbf{D}}_s = (1-\varphi)\widetilde{\mathbf{D}}_m \qquad\qquad \text{pore isotropy} \qquad (20b)$$

These stress-strain and strain-stress relationships of the porous and solid material will now be used to develop the thermo-poro-structural constitutive law of the porous material. It is interesting to note the relationships between $\widetilde{\mathbf{D}}_m$ and $\widetilde{\mathbf{D}}_s$ and between $\widetilde{\mathbf{C}}_m$ and $\widetilde{\mathbf{C}}_s$ are given by the nondimensional diagonal weight vector $\lceil r \rfloor = \lceil \mathbf{I} - \mathbf{W}\rfloor/(1-\varphi)$. In reality, the properties of the solid matrix and the porous matrix are quite different from the relationships outlined in equations (19) and (20), and must be determined by experiment individually. This is caused partially by the nature of the tortuous paths of the pores, and the associated stress concentration factors with these irregular shapes, etc.

Thermo-Poro-Structural Constitutive Law

The following relationships will be developed for the *total* stress and strain variables, with it being understood that the parallel development for the corresponding *incremental* variables follows the same form.

The loads on the representative control volume (RCV) are assumed to be decomposed into separate load sets, where each type of load is applied independently from the others. Towards this end, the *average* mechanical stress and the pore pressure will be used, since the decomposition expressed by equation (10) separates these two loads such that the pore pressure is hydrostatic throughout, i.e., both on the inside boundary and the outer boundary, constituting a single hydrostatic pressure p on the solid skeleton within the RCV. The average mechanical stress is the *non-pressure* component of the total stress acting on the outer boundary of the RCV. This particular stress (force) decomposition permits the strain response to be equally decomposed into two components, one associated with the hydrostatic pressure p, called the pressure strain ϵ^{pr} and the other associated with the *average* mechanical stress σ^a called the mechanical strain $\epsilon^a (= \epsilon^m)$. In addition to the average mechanical stress and pore pressure, there is a thermal strain ϵ^{th} associated with a temperature change and there could possibly be a creep strain ϵ^c

associated with stress and strain loading rates. When the deformations are infinitesimal (the case being dealt with here), the total strain may be expressed as the sum of the mechanical, pressure, thermal and creep strains by

$$\epsilon = \epsilon^m(\sigma^a) + \epsilon^{pr}(p) + \epsilon^{th}(T) + \epsilon^c(T,t) \qquad (21)$$

where the notation (σ^a), (p), (T) and (T,t) indicates that these strains are functions of mechanical stress, pressure, temperature and time. The reason these strains may be added together is that infinitesimal strains are related *linearly* to the displacements, and the total displacements are just the sum of the incremental displacements caused by the different loads (mechanical stress, presure, temperature, etc.).

The relationships between $\sigma^a, \epsilon^a, \sigma^m$, and ϵ^m were given by equations (13) through (19). To complete the thermo-poro-structural derivation, expressions are needed for the pressure strain ϵ^{pr} and the thermal strain ϵ^{th}. One of the more important points to be emphasized here is that the pressure and the temperature act *uniformly on the solid skeleton* within the control volume, i.e., on the solid volume within the RCV. They also act equally on the gas within the pore volume within the RCV, but we are presently interested in the solid response. This is easily seen for pressure in Figures 2 and 3. Therefore, the constitutive relationships between pressure strain and pressure, and thermal strain and temperature only involve the properties of the solid skeleton. With this in mind, and referencing equation (17b) above, the pressure strain and thermal strain are defined by

$$\epsilon^{pr}(p) = -\widetilde{\mathbf{D}}_s \, \delta \, p \qquad (22a)$$

$$\epsilon^{th}(T) = \widetilde{\alpha}_s \, T \qquad (22b)$$

where $\widetilde{\alpha}_s$ are the *total* coefficients of linear thermal expansion of the solid phase of the material, and are related to the instantaneous form α_s via

$$\widetilde{\alpha}_s \, T = \int_0^T \alpha_s(T)\, dT \qquad (22c)$$

Also, since pressure was defined as positive in compression, the minus sign occurs in equation (22a) since all of the strains are, by definition, positive when the displacements are positive. The pressure p and temperature T in equations (22) represents the pressure *above* some reference pressure p_o and the temperature T *above* some reference temperature T_o. Solving for the mechanical strain $\epsilon^m = \epsilon^a$ in equation (21), and substituting the resulting expression into equation (16b), and then substituting

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definitions (22) for ϵ^{pr} and ϵ^{th} into this equation yields the desired constitutive law for the thermo-poro-structural relationship, namely

$$\boxed{\sigma^a = \widetilde{\mathbf{C}}_m (\epsilon + \widetilde{\mathbf{D}}_s \delta p - \alpha_s T - \epsilon^c)} \qquad (23a)$$

The corresponding *incremental* form of this equation is

$$d\sigma^a = \mathbf{C}_m (d\epsilon + \mathbf{D}_s \delta dp - \alpha_s dT - d\epsilon^c) \qquad (23b)$$

Dividing through equation (27b) by an incremental time dt yields the corresponding *rate* from, namely

$$\boxed{\frac{\partial \sigma^a}{\partial t} = \mathbf{C}_m \left(\frac{\partial \epsilon}{\partial t} + \mathbf{D}_s \delta \frac{\partial p}{\partial t} - \alpha_s \frac{\partial T}{\partial t} - \frac{\partial \epsilon^c}{\partial t} \right)} \qquad (23c)$$

The interesting point to observe in these equations is the presence of mechanical properties for both the porous material $(\mathbf{C}_m)$ and the solid phase of the material $(\mathbf{D}_s, \alpha_s)$, both of which may be nonlinear (in which case, the incremental or rate form (23b) or (23c) would have to be used).

Effective Stress Concept

Before leaving this discussion of different *stress measures*, a word must be made concerning what is commonly referred to in the literature as the **effective stress**. The effective stress appears to take many different forms, and therefore seems confusing. For the present derivation, its definition still needs to be determined. Since the theory and the resulting equations do not depend upon what form the effective stress takes, this question may be left open for the present. Remember, the definition of this *pseudo variable* is an artifact, and has nothing to due with the physics. It was introduced into the poro-elasticity formulation to simplify the governing equations, and to allow analytical solutions from the field of solid elasticity theory to be used for solutions to poro-elasticity problems. Its definition is usually given by (e.g., see References [16] or [19])

> The effective stress concept is that the response of a **saturated** porous material may be described by the response law for the **dry** porous material when the applied stress is replaced by the effective stress.

Following this concept for the definition of the *effective stress*, equation (23a) is rewritten as

$$\sigma^a - \widetilde{\mathbf{C}}_m \widetilde{\mathbf{D}}_s \delta p = \widetilde{\mathbf{C}}_m (\epsilon - \alpha_s T - \epsilon^c)$$

Now, all of the terms on the left hand side of this equation may be defined as the **effective stress** σ^e, thereby reducing this equation to an equivalent thermo-elasticity equation (consult any textbook on thermo-elasticity theory), i.e.,

$$\sigma^e = \sigma^a - \widetilde{\mathbf{C}}_m \widetilde{\mathbf{D}}_s \delta p \qquad (24a)$$

$$\sigma^a = \sigma^e + \widetilde{\mathbf{C}}_m \widetilde{\mathbf{D}}_s \delta p \qquad (24b)$$

When substituting equation (24b) into the *average* mechanical stress definition as expressed in equation (10), the following results.

$$\sigma = \sigma^e - [\mathbf{I} - \widetilde{\mathbf{C}}_m \widetilde{\mathbf{D}}_s] \delta p$$

where $\lceil \mathbf{I} \rfloor$ is the unit diagonal matrix (see equations (3)). Therefore, the stress definition given by this equation defines yet another parameter, called the *anisotropic effective stress parameter(s)* κ given by

$$[\kappa] = [\mathbf{I} - \widetilde{\mathbf{C}}_m \widetilde{\mathbf{D}}_s] \qquad (25a)$$

Using definition (25a), the effective stress equation becomes

$$\sigma = \sigma^e - [\kappa] \delta p \qquad (25b)$$

Therefore, equations (25a) and (25b) define the *effective stress* σ^e as used in the present derivation. This definition is the same as that given by Carrol [19] and by Carrol & Katsube [22]. When the material is isotropic, then the anisotropic effective stress parameter κ becomes (see Appendix A for the relationships between anisotropic and isotropic properties)

$$[\kappa] = \left(1 - \frac{k_m}{k_s} \right) \lceil \mathbf{I} \rfloor \qquad (25c)$$

where k_m and k_s are the *bulk moduli* of the porous material and the solid skeleton, respectively. This form (of the parameter κ) was also given by Biot & Willis [15] and later by Nur & Byerlee [16]. This form does not involve the famous Terzaghi effective

stress (see [10] and [11]). The particular role played by the effective stress definition given in equations (25) has yet to be shown, and will not be elaborated upon further here.

This concludes the derivation of the thermo-poro-structural equations governing the deformational characteristics of a porous material loaded by mechanical stresses, pore pressure and thermal gradients.

Nonlinear Pyrolysis Diffusion Equations

The *equation of continuity of mass* which governs the generation and flow of the pyrolysis gas within an ablating nozzle liner material is given by

$$\underbrace{\frac{\partial m_g}{\partial t}}_{\text{storage}} + \underbrace{\nabla \cdot (\tilde{\mathbf{m}}_g)}_{\text{diffusion}} = \underbrace{\frac{\partial \hat{m}_g}{\partial t}}_{\text{generation}} \tag{26}$$

where m_g is the mass of the gas in the representative control volume (RCV), $\tilde{\mathbf{m}}_g$ is the gas mass flux, $\hat{m}_g$ is the mass of the gas generated due to pyrolysis and t is the time variable. The first term represents the rate at which gas mass is stored within the RCV. The second term represents the net flux of gas entering and leaving the RCV (per unit time). The sum of these two terms must equal the last term, the rate at which gas is being generated within the RCV. The three independent variables used to describe the dependence of m_g are gas pressure p, gas (and solid) temperature T, and strain ϵ. Therefore, $m_g = m_g(p, T, \epsilon)$. The rate of gas generation $\hat{m}_g$ is a function of the "*extent of pyrolysis*" or "*degree of char*" β, which varies from 0 (virgin material) to 1 (fully charred material), and will be explained below. Each of the three terms in equation (26) will be developed next.

Mass Storage Term

The mass of the gas m_g in a deforming RCV is given, by definition, as the density of the gas ρ_g times the volume occupied by the gas, namely, the pore volume V_p, i.e.,

$$m_g = \rho_g V_p \tag{27}$$

Therefore, the time rate of change of m_g is given by

$$\frac{\partial m_g}{\partial t} = V_p \frac{\partial \rho_g}{\partial t} + \rho_g \frac{\partial V_p}{\partial t} = \rho_g V \left[\varphi \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial t} + \frac{1}{V} \frac{\partial V_p}{\partial t} \right] \tag{28}$$

where equation (1b) which relates the instantaneous pore volume V_p to the instantaneous total volume V and porosity φ has been used. The main task in the present derivation is to determine the time rate of change of the pore volume V_p, especially in the presence of pressure p, temperature T, and mechanical strain ϵ. The main coupling between pore pressure and mechanical deformations exists through this term. Consider the condition where pressure builds up resulting in a tendency to open up the porosity and permeability, which in turn allows the gas flow to increase and the pressure to decrease. An equilibrium condition between pressure and porosity/permeability exists somewhere between these two states.

Taking the time derivative of total volume as expressed in equation (1a) and solving for $\partial V_p / \partial t$ yields

$$\frac{\partial V_p}{\partial t} = \frac{\partial V}{\partial t} - \frac{\partial V_s}{\partial t} \tag{29}$$

However, the present derivation treats the strain ϵ in the RCV as an independent variable (along with the pressure p and temperature T). Therefore, if expressions are generated for the time rate of change of the total volume V and the solid volume V_s in terms of their respective *dilatational* strain rates, then the time rate of change of the pore volume will be known.

Any textbook on solid mechanics will show that for infinitesimal strain definitions, the change in the total volume of the representative control volume (RCV) may be related to the dilatational (or volumetric) strain e_v of that total volume by

$$V = V_o + \delta V = V_o (1 + e_v) = V_o \left(1 + \delta^T \epsilon \right) \tag{30}$$

where ϵ is the total strain vector (of the total control volume, i.e., the RCV) and δ is the Kronecker delta operation which simply sums the direct strain components of ϵ (see equations (3)). Also, V_o is the initial total volume and is considered constant (for purposes of taking derivatives). Taking the time derivative of the total volume as expressed in equation (30) yields

$$\frac{\partial}{\partial t}(V) = \frac{\partial}{\partial t} \left[V_o \left(1 + \delta^T \epsilon \right) \right] = V_o \delta^T \frac{\partial \epsilon}{\partial t} \tag{31}$$

Therefore, the *only* change of the total volume is due to the mechanical straining of that volume. Any changes due to chemical or pyrolysis effects occur *within* the RCV thereby preserving the *total* volume.

The solid phase of the material has similar mechanical straining relationships to those presented above for the total volume, with the only difference being the *reference volume measure*. However, there is an additional *pyrolysis event* that contributes to the time rate of change of the solid volume V_s. For the purpose of determining the time rate of change of the solid volume, the *pyrolysis event* will be briefly discussed, with the bulk of this discussion to be covered in detail later.

When pyrolysis occurs, the solid gives up mass (and volume) to the gas. The independent variable used to measure the "*extent of pyrolysis*" is β, as mentioned in the discussion following equation (26). At the same time that the solid volume V_s decreases (due to pyrolysis mass loss), an equal and opposite increase occurs in the pore volume V_p, thereby preserving the total volume within the RCV. This may be expressed by

$$-\frac{\partial V_s^\beta}{\partial t} = \frac{\partial V_p^\beta}{\partial t} \tag{32}$$

where the superscript β refers to the dependence upon the extent of pyrolysis. Therefore, this term will be included in the current derivation for the time rate of change of the solid volume V_s, but will not be developed until later.

With this in mind, the total change in the solid volume may be viewed as the sum of a mechanical straining part and a pyrolysis part. The mechanical straining part is found by repeating definitions (30) and (31) for the solid volume, with $e_v^s = \delta^T \epsilon_s$ being the dilatational strain in the solid skeleton and V_{so} being the initial solid volume. The pyrolysis part is given by the term V_s^β. The instantaneous solid volume V_s is therefore given by

$$V_s = V_{so} + \delta V_s^\epsilon + V_s^\beta = V_{so}\left(1 + \delta^T \epsilon_s\right) + V_s^\beta$$

Taking the time derivative of the solid volume as expressed by the above equation yields

$$\frac{\partial}{\partial t}\left(V_s\right) = V_{so}\delta^T \frac{\partial \epsilon_s}{\partial t} + \frac{\partial V_s^\beta}{\partial t}$$

However, the original solid volume V_{so} is related to the original total volume V_o by equation (1c). Making the appropriate substitution in the above equation yields

$$\frac{\partial}{\partial t}\left(V_s\right) = V_o(1-\varphi)\delta^T \frac{\partial \epsilon_s}{\partial t} + \frac{\partial V_s^\beta}{\partial t} \tag{33}$$

Substituting equations (31) and (33) into equation (29), and replacing the time rate of change of the solid volume with the time rate of change of the pore volume (as per equation (32)) leads to

$$\frac{\partial V_p}{\partial t} = V_o \delta^T \left[\frac{\partial \epsilon}{\partial t} - (1-\varphi)\frac{\partial \epsilon_s}{\partial t}\right] + \frac{\partial V_p^\beta}{\partial t} \tag{34}$$

Since the total strain ϵ is an independent variable, the strain in the solid phase of the material ϵ_s must now be expressed in terms of the total strain ϵ (and also p and T).

As stated earlier (see the discussion before equation (21)), the mechanical stress σ^a, pore pressure p and temperature T loads on the solid phase of the material may be applied independently from one another. Remember, equation (10) implies that the pressure component of load is applied to *all of the solid skeleton boundaries*, both internal and external. The average mechanical stress σ^a is the non-pressure component of the total load and is applied only to the outer boundary of the RCV. With this in mind, then the total strain within the solid phase of the material may be given in total strain form and strain rate form by

$$\epsilon_s = \epsilon_s^\sigma + \epsilon_s^{pr} + \epsilon_s^{th} \tag{35a}$$

$$\frac{\partial \epsilon_s}{\partial t} = \frac{\partial \epsilon_s^\sigma}{\partial t} + \frac{\partial \epsilon_s^{pr}}{\partial t} + \frac{\partial \epsilon_s^{th}}{\partial t} \tag{35b}$$

The strain ϵ_s^σ due to the average mechanical stress σ^a is given by ϵ^a, i.e., the strain in the solid skeleton. Therefore, using the first of equations (14) to express ϵ^a in terms of ϵ^m, and equation (17b) to express ϵ^m in terms of σ^m, and finally equation (13) to express σ^m in terms of σ^a leads to (for the pore anisotropy case)

$$\epsilon_s^\sigma = \epsilon^a = (1-\varphi)\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\epsilon^m = (1-\varphi)\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\tilde{\mathbf{D}}_s\,\sigma^m = (1-\varphi)\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\tilde{\mathbf{D}}_s\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\sigma^a$$

or when expressed in rate form, this equation becomes (for both pore geometry cases)

$$\frac{\partial \epsilon_s^\sigma}{\partial t} = (1-\varphi)\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\mathbf{D}_s\lceil \mathbf{I} - \mathbf{W}\rfloor^{-1}\frac{\partial \sigma^a}{\partial t} \qquad \text{pore anisotropy}$$

$$\frac{\partial \epsilon_s^\sigma}{\partial t} = \frac{1}{(1-\varphi)}\mathbf{D}_s\frac{\partial \sigma^a}{\partial t} \qquad \text{pore isotropy} \tag{36}$$

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