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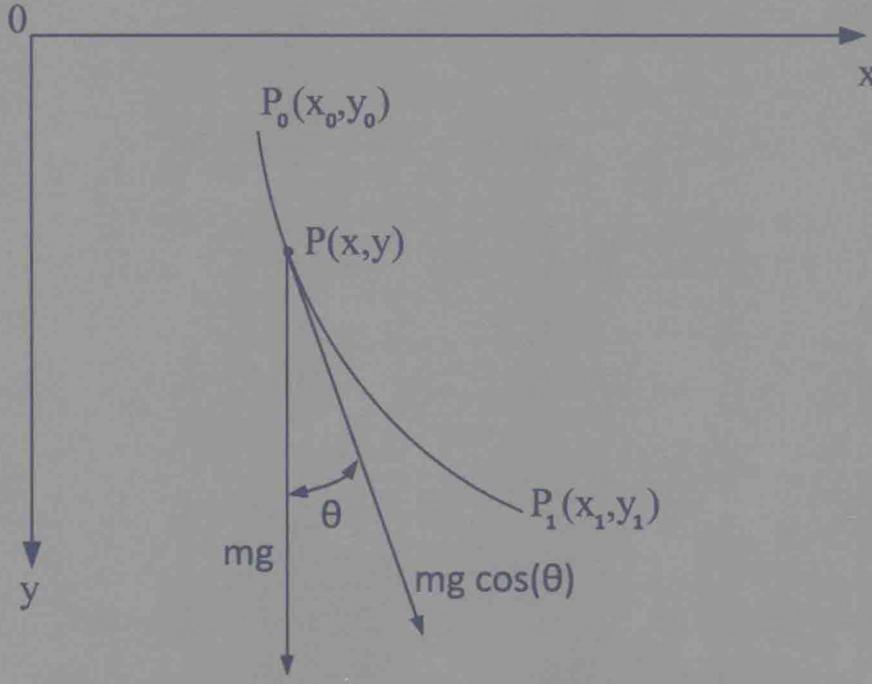

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CRC Press
Taylor & Francis Group

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CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
Version Date: 20120716

International Standard Book Number: 978-1-4665-6026-0 (Hardback)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

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Nonlinear Optimal Control Theory

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
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A CHAPMAN & HALL BOOK

Foreword

This book provides a thorough introduction to optimal control theory for nonlinear systems. It is a sequel to Berkovitz's 1974 book entitled *Optimal Control Theory.* In optimal control theory, the Pontryagin principle, Bellman's dynamic programming method, and theorems about existence of optimal controls are central topics. Each of these topics is treated carefully. The book is enhanced by the inclusion of many examples, which are analyzed in detail using Pontryagin's principle. These examples are diverse. Some arise in such applications as flight mechanics, and chemical and electrical engineering. Other examples come from production planning models and the classical calculus of variations.

An important feature of the book is its systematic use of a relaxed control formulation of optimal control problems. The concept of relaxed control is an extension of L. C. Young's notion of generalized curves, and the related concept of Young measures. Young's pioneering work in the 1930s provided a kind of "generalized solution" to calculus of variations problems with nonconvex integrands. Such problems may have no solution among ordinary curves. A relaxed control, as defined in Chapter 3, assigns at each time a probability measure on the space of possible control actions. The approach to existence theorems taken in Chapters 4 and 5 is to prove first that optimal relaxed controls exist. Under certain Cesari-type convexity assumptions, optimal controls in the ordinary sense are then shown to exist. The Pontryagin maximum principle (Chapters 6 and 7) provides necessary conditions that a relaxed or ordinary control must satisfy. In the relaxed formulation, it turns out to be sufficient to consider discrete relaxed controls (see Section 6.3). This is a noteworthy feature of the authors' approach.

In the control system models considered in Chapters 2 through 8, the state evolves according to ordinary differential equations. These models neglect possible time delays in state and control actions. Chapters 10, 11, and 12 consider models that allow time delays. For "hereditary systems" as defined in Chapter 10, Pontryagin's principle takes the form in Theorem 10.3.1. Hereditary control problems are effectively infinite dimensional. As explained in Section 10.6, the true state is a function on a time interval $[-r, 0]$ where r represents the maximum time delay in the control system. Chapter 11 considers hereditary system models, with the additional feature that states are constrained by given bounds. For readers interested only in control systems without time delays, necessary conditions for optimality in bounded state problems are described in Section 11.6.

The dynamic programming method leads to first order nonlinear partial differential equations, which are called Hamilton-Jacobi-Bellman equations (or sometimes Bellman equations). Typically, the value function of an optimal control problem is not smooth. Hence, it satisfies the Hamilton-Jacobi-Bellman equation only in a suitable "generalized sense." The Crandall-Lions Theory of viscosity solutions provides one such notion of generalized solutions for Hamilton-Jacobi-Bellman equations. Work of A. I. Subbotin and co-authors provides another interesting concept of generalized solutions. Chapter 12 provides an introduction to Hamilton-Jacobi Theory. The results described there tie together in an elegant way the viscosity solution and Subbotin approaches. A crucial part of the analysis involves a lower Dini derivate necessary condition derived in Section 12.4.

The manuscript for this book was not quite in final form when Leonard Berkovitz passed away unexpectedly. He is remembered for his many original contributions to optimal control theory and differential games, as well as for his dedicated service to the mathematics profession and to the control community in particular. During his long career at Purdue University, he was a highly esteemed teacher and mentor for his PhD students. Colleagues warmly remember his wisdom and good humor. During his six years as Purdue Mathematics Department head, he was resolute in advocating the department's interests. An obituary article about Len Berkovitz, written by W. J. Browning and myself, appeared in the January/February 2010 issue of *SIAM News*.

Wendell Fleming

Preface

This book is an introduction to the mathematical theory of optimal control of processes governed by ordinary differential and certain types of differential equations with memory and integral equations. The book is intended for students, mathematicians, and those who apply the techniques of optimal control in their research. Our intention is to give a broad, yet relatively deep, concise and coherent introduction to the subject. We have dedicated an entire chapter to examples. We have dealt with the examples pointing out the mathematical issues that one needs to address.

The first six chapters can provide enough material for an introductory course in optimal control theory governed by differential equations. Chapters 3, 4, and 5 could be covered with more or less details in the mathematical issues depending on the mathematical background of the students. For students with background in functional analysis and measure theory, Chapter 7 can be added. Chapter 7 is a more mathematically rigorous version of the material in Chapter 6.

We have included material dealing with problems governed by integrodifferential and delay equations. We have given a unified treatment of bounded state problems governed by ordinary, integrodifferential, and delay systems. We have also added material dealing with the Hamilton-Jacobi Theory. This material sheds light on the mathematical details that accompany the material in Chapter 6.

The material in the text gives a sufficient and rigorous treatment of finite dimensional control problems. The reader should be equipped to deal with other types of control problems such as problems governed by stochastic differential equations and partial differential equations, and differential games.

I am very grateful to Mrs. Betty Gick of Purdue University and Mrs. Annette Rohrs of Georgia Institute of Technology for typing the early and final versions of the book. I am very grateful to Professor Wendell Fleming for reading the manuscript and making valuable suggestions and additions that improved and enhanced the quality of the book as well as avoided and removed errors. I also wish to thank Professor Boris Mordukovich for reading the manuscript and making valuable suggestions. All or parts of the material up to the first seven chapters have been used for optimal control theory courses in Purdue University and North Carolina State University.

This book is a sequel to the book *Optimal Control Theory* by Leonard

D. Berkovitz. I learned control theory from this book taught by him. We decided to write the current book in 1994 and we went through various versions.

L. D. Berkovitz was my teacher and a second father. He passed away on October 13, 2009 unexpectedly. He was caring, humble, and loved mathematics. He is missed greatly by all who were fortunate enough to have known him. This book was completed before his death.

Negash G. Medhin
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Contents

Chapter 1

Examples of Control Problems

1.1 Introduction

Control theory is a mathematical study of how to influence the behavior of a dynamical system to achieve a desired goal. In optimal control, the goal is to maximize or minimize the numerical value of a specified quantity that is a function of the behavior of the system. Optimal control theory developed in the latter half of the 20th century in response to diverse applied problems. In this chapter we present examples of optimal control problems to illustrate the diversity of applications, to raise some of the mathematical issues involved, and to motivate the mathematical formulation in subsequent chapters. It should not be construed that this set of examples is complete, or that we chose the most significant problem in each area. Rather, we chose fairly simple problems in an effort to illustrate without excessive complication.

Mathematically, optimal control problems are variants of problems in the calculus of variations, which has a 300+ year history. Although optimal control theory developed without explicit reference to the calculus of variations, each impacted the other in various ways.

1.2 A Problem of Production Planning

The first problem, taken from economics, is a resource allocation problem; the Ramsey model of economic growth. Let $Q(t)$ denote the rate of production of a commodity, say steel, at time t. Let $I(t)$ denote the rate of investment of the commodity at time t to produce capital; that is, productive capacity. In the case of steel, investment can be thought of as using steel to build new steel mills, transport equipment, infrastructure, etc. Let $C(t)$ denote the rate of consumption of the commodity at time t. In the case of steel, consumption can be thought of as the production of consumer goods such as automobiles. We assume that all of the commodity produced at time t must be allocated to either investment or consumption. Then

$$Q(t) = I(t) + C(t) \qquad I(t) \geq 0 \quad C(t) \geq 0.$$

We assume that the rate of production is a known function F of the capital at time t. Thus, if $K(t)$ denotes the capital at time t, then

$$Q(t) = F(K(t)),$$

where F is a given function. The rate of change of capital is given by the capital accumulation equation

$$\frac{dK}{dt} = \alpha I(t) - \delta K(t) \quad K(0) = K_0, \; K(t) \geq 0,$$

where the positive constant α is the growth rate of capital and the positive constant δ is the depreciation rate of capital. Let $0 \leq u(t) \leq 1$ denote the fraction of production allocated to investment at time t. The number $u(t)$ is called the savings rate at time t. We can therefore write

$$\begin{aligned} I(t) &= u(t)Q(t) = u(t)F(K(t)) \\ C(t) &= (1 - u(t))Q(t) = (1 - u(t))F(K(t)), \end{aligned}$$

and

$$\begin{aligned} \frac{dK}{dt} &= \alpha u(t)F(K(t)) - \delta K(t) \\ K(t) &\geq 0 \quad K(0) = K_0. \end{aligned} \tag{1.2.1}$$

Let $T > 0$ be given and let a "social utility function" U, which depends on C, be given. At each time t, $U(C(t))$ is a measure of the satisfaction society receives from consuming the given commodity. Let

$$J = \int_0^T U(C(t))e^{-\gamma t}\, dt,$$

where γ is a positive constant. Our objective is to maximize J, which is a measure of the total societal satisfaction over time. The discount factor $e^{-\gamma t}$ is a reflection of the phenomenon that the promise of future reward is usually less satisfactory than current reward.

We may rewrite the last integral as

$$J = \int_0^T U((1 - u(t))F(K(t)))e^{-\gamma t} dt. \tag{1.2.2}$$

Note that by virtue of (1.2.1), the choice of a function $u \colon [0, T] \to u(t)$, where u is subject to the constraint $0 \leq u(t) \leq 1$ determines the value of J. We have here an example of a *functional*; that is, an assignment of a real number to every function in a class of functions. If we relabel K as x, then the problem of maximizing J can be stated as follows:

Choose a savings program u over the time period $[0, T]$, that is, a function u defined on $[0, T]$, such that $0 \leq u(t) \leq 1$ and such that

$$J(u) = -\int_0^T U((1-u(t))F(\varphi(t)))e^{-\gamma t}dt \tag{1.2.3}$$

is minimized, where φ is a solution of the differential equation

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x \quad \varphi(0) = x_0,$$

and φ satisfies $\varphi(t) \geq 0$ for all t in $[0, T]$. The problem is sometimes stated as

Minimize:

$$J(u) = -\int_0^T U((1-u(t))F(x))e^{-\gamma t}dt$$

Subject to:

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x, \quad x(0) = x_0, \quad x \geq 0, \quad 0 \leq u(t) \leq 1$$

1.3 Chemical Engineering

Let $x^1(t), \ldots, x^n(t)$ denote the concentrations at time t of n substances in a reactor in which n simultaneous chemical reactions are taking place. Let the rates of the reactions be governed by a system of differential equations

$$\frac{dx^i}{dt} = G^i\left(x^1, \ldots, x^n, \theta(t), p(t)\right) \qquad x^i(0) = x_0^i \quad i = 1, \ldots, n. \tag{1.3.1}$$

where $\theta(t)$ is the temperature in the reactor at time t and $p(t)$ is the pressure in the reactor at time t. We control the temperature and pressure at each instance of time, subject to the constraints

$$\begin{aligned} \theta_b &\leq \theta(t) \leq \theta_a \\ p_b &\leq p(t) \leq p_a \end{aligned} \tag{1.3.2}$$

where θ_a, θ_b, p_a, and p_b are constants. These represent the minimum and maximum attainable temperature and pressure.

We let the reaction proceed for a predetermined time T. The concentrations at this time are $x^1(T), \ldots, x^n(T)$. Associated with each product is an economic value, or price c^i, $i = 1, \ldots, n$. The price may be negative, as in the case of hazardous wastes that must be disposed of at some expense. The value of the end product is

$$V(p, \theta) = \sum_{i=1}^{n} c^i x^i(T). \tag{1.3.3}$$

Given a set of initial concentrations x_0^i, the value of the end product is completely determined by the choice of functions p and θ if the functions G^i have certain nice properties. Hence the notation $V(p, \theta)$. This is another example of a functional; in this case, we have an assignment of a real number to each pair of functions in a certain collection.

The problem here is to choose piecewise continuous functions p and θ on the interval $[0, T]$ so that (1.3.2) is satisfied and so that $V(p, \theta)$ is maximized.

A variant of the preceding problem is the following. Instead of allowing the reaction to proceed for a fixed time T, we stop the reaction when one of the reactants, say x^1, reaches a preassigned concentration x_f^1. Now the final time t_f is not fixed beforehand, but is the smallest positive root of the equation $x^1(t) = x_f^1$. The problem now is to maximize

$$V(p, \theta) = \sum_{i=2}^{n} c^i x^i(t_f) - k^2 t_f.$$

The term $k^2 t_f$ represents the cost of running the reactor.

Still another variant of the problem is to stop the reaction when several of the reactants reach preassigned concentrations, say $x^1 = x_f^1$, $x^2 = x_f^2, \ldots, x^j = x_f^j$. The value of the end product is now

$$\sum_{i=j+1}^{n} c^i x^i(t_f) - k^2 t_f.$$

We remark that in the last two variants of the problem there is another question that must be considered before one takes up the problem of maximization. Namely, can one achieve the desired final concentrations using pressure and temperature functions p and θ in the class of functions permitted?

1.4 Flight Mechanics

In this problem a rocket is taken to be a point of variable mass whose moments of inertia are neglected. The motion of the rocket is assumed to take place in a plane relative to a fixed frame. Let $y = (y^1, y^2)$ denote the position vector of the rocket and let $v = (v^1, v^2)$ denote the velocity vector of the rocket. Then

$$\frac{dy^i}{dt} = v^i \qquad y^i(0) = y_0^i \qquad i = 1, 2, \tag{1.4.1}$$