Theory of Technical Change and Economic Invariance

Application of Lie Groups

RYUZO SATO

ECONOMIC THEORY, ECONOMETRICS, AND MATHEMATICAL ECONOMICS

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Application of Lie Groups

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To
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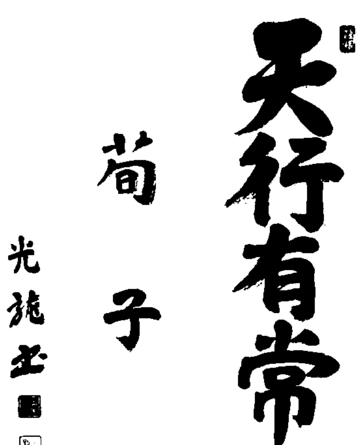

"Nature's ways are invariant"
Hsün Tzu
(ca. 305–235 B.C.)
CALLIGRAPHY BY REVEREND ROSHI KORYU

FOREWORD

Mathematics is the handmaiden of the sciences. But mathematics also has a life of her own, gaining as much in her own development and fulfillment from the sciences as she gives to them. To help describe how apples and planets fall, and how ropes hang, Newton and Leibniz developed the calculus. By serendipity, that mode of analysis permitted economists to perfect the theory of general equilibrium two centuries later.

Sometimes the logical tools are forged long before their practical use becomes apparent. The calculus of variations was formulated in the eighteenth century and, in its classical form, perfected a century ago. Lying dormant for decades, it was brought back to life by the kiss of the dynamic programmer in pursuit of optimal control. Quaternions were something of an anticlimax until the spin matrices of quantum mechanics brought them back into vogue. Albert Einstein's quest for a general theory of relativity was helped by the fortuitous earlier development of the Ricci tensor calculus. Applied researchers even reinvent the wheel, as when Werner Heisenberg's rules for combining the elements in the array of quantum observables turned out to mimic the rules for matrix multiplication. Probability theory would remain a rather dry branch of measure theory were it not for its usefulness in describing how dice fall, multitudes die, and atoms collide.

Barely a century ago, the Norwegian Sophus Lie developed the theory of what have come to be called *Lie transformation groups*. Their original primary application was to the classical mechanics of Lagrange and Hamilton. Now Ryuzo Sato of Brown University is making a pioneering attempt to apply the Lie theory to modern economics. Here are only a few of his explorations.

1. It is popular to speak of ''labor-saving inventions,'' which enable nine men to do the work of ten. This is the special case of a ''factor-augmenting invention,'' in which land or labor or any input becomes equivalent in efficiency to a multiple of itself. Dr. Sato treats such technical processes, and more general ones, as examples of Lie groups.

2. I once posed the open question: What demand functions are *self-dual,* in the sense that their natural duals have *exactly the same mathematical form as themselves?* (The unit-elastic Cobb–Douglas case, in which any good's relative expenditure $p_i q_i / I = k_i$, is an obvious case; but what other cases are there in which

$$\mathbf{q} = \mathbf{f}(\mathbf{y}), \qquad \mathbf{y} = \mathbf{f}(\mathbf{q}),$$

$$\mathbf{q} = [q_1, \ldots, q_n], \qquad \mathbf{y} = [p_1/\sum_1^n p_j q_j, \ldots, p_n/\sum_1^n p_j q_j],$$

and $\mathbf{f}(\mathbf{y}) = [f_1(\mathbf{y}), \ldots, f_n(\mathbf{y})]$?) Dr. Sato, using the methods of Lie groups, has extended the answers given by Professor Wahidul Haque and others.

3. Here is the final example from a long list of possible examples. Along an intertemporally efficient path of a *closed* von Neumann system, in which every good of a set of goods and its rate of growth are producible out of those *same goods as inputs* by a *constant-returns-to-scale* technology, the ratio of the total value of the capital goods to total income is a fundamental constant. Dr. Sato now shows that this 1970 finding of mine is essentially the only ''energy integral'' that such a dynamic system can in general possess.

I believe that Ryuzo Sato is the only scholar who has studied the application of Lie groups to economics. Not until economists have given his impressive treatise a thorough reading shall we be in a position to put useful bounds on the value added to economics by this powerful and elegant technique. The ball is now in our court.

Massachusetts Institute of Technology PAUL A. SAMUELSON

PREFACE

This book is based on lectures delivered at Brown University over a period of several years and at the University of Bonn, West Germany, during the academic year of 1974. It deals with a variety of topics in economic theory, ranging from the analysis of production functions to the general recoverability problem of optimal dynamic behavior. However, I treat the various selected topics, which interest me, from the unifying point of view of "transformation and invariance." In general, the book is concerned with the economic invariance problems of observable behavior under general transformations such as technical change and/or taste change. It is fundamentally a study of market behavior and economic invariance under "Lie types of technical change" (the exact definition being given in the text).

I became interested in the area of transformation and invariance through my continued involvement in the study of the theory of technical change. My initial contact with Lie group theory, however, goes back to my days at Johns Hopkins University in the early 1960s, when I was exploring different branches of mathematics, including differential geometry, for enjoyment. I was then unaware of any economic relevancy of the theory. The realization of the usefulness of this aspect of mathematics in economic theory came much later when I studied, as a Guggenheim Fellow, the physical applications of Lie groups. Samuelson's short note [1970] "Law of Conservation of the Capital–Output Ratio" (see Chapter 7 for the exact reference) was the catalyst that inspired my perception of the direct link between optimal economic behavior and Lie groups. Although Samuelson's article itself is not directly related to Lie groups, I realized that conservation laws can be deduced from the invariance properties of optimal dynamic behavior under (Lie group) transformations by application of Noether's theorem (see Chapter 7). This initial insight led me to the application of Lie group theory to other areas presented in the book.

This book is intended for economists, and hence emphasis is placed on economic interpretation rather than mathematical rigor. The book assumes a knowledge of the basic elements of modern economic theory as well as some amount of elementary mathematics used in economics. Beyond this it is self-contained: The reader who is not familiar with even the elementary aspects of Lie's theory of transformation groups can read this book by first studying the brief survey of Lie group theory presented in the Appendix. Although this book is not meant to be a textbook, it is hoped that the book may be used for advanced undergraduate courses and for graduate courses in economics. Specialists in applied mathematics and natural sciences may also find it useful, especially for learning the manner in which the same methodology is consistently applied in theoretical economics as in other branches of modern scientific endeavors.

The writing of this book was a long and arduous task, and I received help from many people. My greatest debt is to Paul A. Samuelson, who, in one way or another, is responsible for some of the topics discussed. His inspiration and encouragement are in evidence throughout the volume. I owe my deepest appreciation to Takayuki Nôno, who read the entire manuscript in various stages and offered numerous suggestions for its improvement. His careful criticism has saved me from many errors, and his advice and influence are reflected throughout the book.

I am grateful to the former and present members of the Mathematical Economics Workshop at Brown University, notably to Martin J. Beckmann, Yannis M. Ioannides, Allan M. Feldman, Gilbert Suzawa (in addition to his editing), Hajime Hori, Rama V. Ramachandran, Hiroshi Ono, Yasuo Kawashima, Philip S. Kott, Joel D. Scheraga, Behzad Diba, Thomas M. Mitchell, Mariko Fujii, Paul Calem, John Rizzo, Paul Segerstrom, Kazuo Mino, and Shun'ichi Tsutsui.

In addition, I have benefited greatly from useful comments and criticisms offered when I presented parts of this book at various universities. My special thanks go to Hendrik Houthakker, Dale Jorgenson, Wilhelm Krelle, Miyohei Shinohara, Karl Shell, Isamu Yamada, Thomas R. Saving, William R. Russell, Akira Takayama, Shujiro Sawada, Seichi Ota, Mineo Ikeda, Hukukane Nikaido, Eiji Ohsumi, Robert L. Basmann, Lawrence J. Lau, P. J. Hammond, Rolf Färe, Michael D. Intriligator, Bryan Ellickson, Yoshimasa Kurabayashi, Shuntaro Shishido, Taro Yamane, Noboru Sakashita, and I-Min Chiang.

The present book includes certain of my previous articles. Thanks are due to the editors of *Review of Economic Studies* and *Econometrica*, and to the Springer-Verlag and the J. C. B. Mohr publishing companies for permission to use my articles which were originally published by them. I gratefully acknowledge the support of the National Science Foundation and of the Guggenheim Foundation in enabling me to pursue and complete the present work.

I express my sincere gratitude to Marion Wathey for a superb typing job with her bionic fingers. Finally, I must express my great thanks to my wife, Kishie, and to my children, Luke (Ryuku) and Elly (Eri), for their patience and encouragement.

Brown University RYUZO SATO

CONTENTS

CHAPTER 1

AN OVERVIEW

I. Introduction: Why Lie Groups?

1. The past 20 years have exhibited a remarkable progress in many branches of economic science. One notable area of research which has attracted the attention of a large number of competent economists is what may be called "the theory of observable market behavior." One of the oldest branches of this approach (started by Samuelson with finishing touches provided by Houthakker) is the theory of revealed preference. It is, in a way, the most direct challenge to the standard theory of consumer behavior, which begins with axiomatic assumptions in order to deduce theorems stating the properties of the consumer's optimal behavior. The theory of revealed preference takes a reverse process of, first, observing rational behavior in the market, and then deriving the theory of consumption and utility analysis consistent with the market behavior.

The theory of observable behavior has recently taken another important turn by the introduction of the "duality analysis" of consumer and firm's behavior. Rather than studying the market behavior generated from the direct (or primal) functions, one begins with the observation of the market behavior related to the indirect (or dual) functions. Here, contrasted with the direct approach, the analysis focuses on the recoverability problem of the expenditure (or indirect utility) function and of the cost function. The properties of the underlying functions such as the utility and production functions are then studied through the recovered indirect functions (see Fuss and McFadden [1978]). This indirect and dual approach not only contributes important insights of its own, but also offers more immediate empirical application. It enables one to formulate many problems in a way that is "natural" (Baumol [1977, p. 354]).

The primary purpose of this book is to develop still further the theory of observable behavior by analyzing the "invariant" relationships among economic variables, often represented by (partial) differential equation systems, by employing a relatively simple aspect of Lie's theory of continuous transformations. It is essential to recognize that the observable market behavior both in the direct and indirect approaches usually manifests itself in *the form of differential equation systems* (often partial). But thus far in the economic literature, very little effort has been given to the study of these differential equations from economic and formal (mathematical) points of view. This book deals with the *economic invariance problems of observable behavior under general "economic transformations" such as "technical change" and "taste change."* It is basically a study of economic invariance under "Lie types of technical change." The title of this volume may be somewhat misleading. This book does *not* deal with every aspect of technical change. Other than presenting a rather general theory of *endogenous* technical progress, the book does not directly deal with the standard problems of technical progress, such as the diffusion process and patent problems.

To demonstrate what is meant by a Lie group and to say why Lie groups are relevant here, let us consider a typical estimation problem of the underlying production function and technical change. Assume that technical progress in the production process is a priori known to have the simple "neutral" form

$$T_t\colon \quad \bar{K} = e^{\alpha t}K, \qquad \bar{L} = e^{\alpha t}L,$$

where K is the capital, L the labor, α the rate of technical progress ($\alpha \geqq 0$), $\bar{K}$ the "effective" capital, $\bar{L}$ the "effective" labor, and t the index of technical progress. The equations for $\bar{K}$ and $\bar{L}$, which may be called the technical progress functions for capital and labor, constitute a *one-parameter Lie group of continuous transformations* (Lie [1891]). Let the parameter of technical progress t change from t_0 to t_1. Then $\bar{K}$ and $\bar{L}$ change from

$$T_{t_0}\colon \quad \bar{K}_0 = e^{\alpha t_0}K, \quad \bar{L}_0 = e^{\alpha t_0}L, \qquad \text{to} \quad T_{t_1}\colon \quad \bar{K}_1 = e^{\alpha t_1}K, \quad \bar{L}_1 = e^{\alpha t_1}L.$$

The technical progress functions constitute a Lie group for the following reasons:

(i) (*Composition*) The result of the successive performance of T_0 and T_1 is the same as that of the single transformation

$$T_{t_2}\colon \quad \bar{K}_2 = \exp(\alpha(t_0 + t_1))K, \qquad \bar{L}_2 = \exp(\alpha(t_0 + t_1))L.$$

(ii) (*Identity*) When there is no technical change $t = 0$, then $\bar{K} = K$ and $\bar{L} = L$.

(iii) (*Inverse*) The inverse functions of T_t are also a member of T, when t is replaced by $-t$,

$$T_t^{-1} = T_{-t}: \quad K = e^{-\alpha t}\bar{K}, \qquad L = e^{-\alpha t}\bar{L}.$$

From the aggregate of the transformation included in the family T_t, where t varies continuously over a given range, any particular transformation of the family is obtained by assigning a particular value to t. Any successive transformations (including identity and inverse transformations) of the family are equivalent to a single transformation of the family. These are the basic properties of a Lie group. (See the Appendix and Chapter 2 for a more precise definition.)

Now assume that the estimation equation is derived from the market observation on the marginal rate of substitution between capital K and labor L by

$$p_K/p_L = Y_K/Y_L = f(K/L, t),$$

where p_K is the price of capital, p_L the price of labor, Y the output, $Y_K = \partial Y/\partial K$ the marginal product of K, and $Y_L = \partial Y/\partial L$ the marginal product of L. If K and L are related with $\bar{K}$ and $\bar{L}$ by the technical progress functions T_t given in the foregoing and if T_t is the only source of technical progress of the system, then it is seen immediately that the estimated marginal rate of substitution f should not contain t, because f coincides with the quantity known as the *invariant* of the group, i.e.,

$$f(K/L, t) = f(\bar{K}/\bar{L}) = f(e^{\alpha t}K/e^{\alpha t}L) = f(K/L).$$

This means that the efficiency increase of capital and labor α cannot be estimated from the observed behavior of the marginal rate of substitution. Furthermore, from the behavior of f, it is "impossible" to identify any "economies of scale" even if they exist. This is because the underlying production function is a member of the so-called *invariant family* of curves generated by this group.

In general, given a Lie type of technical progress T_t, one can always derive a family of production functions invariant under T_t (holothetic technology, see Chapter 2). Conversely, given the observable marginal rate of substitution in the form of a differential equation

$$M(K, L)\,dK + N(K, L)\,dL = 0, \quad \text{or} \quad -\frac{dL}{dK} = \frac{p_K}{p_L} = \frac{M(K, L)}{N(K, L)} = \frac{Y_K}{Y_L},$$

there exists a one-parameter Lie group of transformations (Lie type of technical change) which leaves the underlying production function invariant. If we know beforehand how this type of technical change acts on capital and labor, we can use this knowledge to find the underlying production function and to study its properties. This is an important reason why we may want to study the application of Lie groups.

Consider as another example the case of Shephard's lemma: One can observe the "optimal" production behavior from the factor demand functions

$$x_i = x_i(p, Y), \qquad 1 \leqq i \leqq n,$$

where x_i is the demand for the ith input, p the relative price vector of factor inputs, and Y the output. But in view of Shephard's lemma, the foregoing is equal to

$$\partial C/\partial p_i = x_i(p, Y) = x_i, \qquad 1 \leqq i \leqq n,$$

where C is the total cost function. There are two invariance properties in the preceding equation. First, in order to ascertain that this equation is derived from the *same* underlying cost function $C = C(p, Y)$ for all i, $1 \leqq i \leqq n$, the integrability condition of symmetry must be satisfied for all the x_i. But this is nothing but the *invariant* condition of the partial differential equations, known as the "involution condition" of a Lie group. By writing the factor demand functions in terms of their ratios

$$\frac{\partial C}{\partial p_i}\bigg/\frac{\partial C}{\partial p_j} = \frac{x_i(p, Y)}{x_j(p, Y)} = R^{ij}(p, Y), \qquad 1 \leqq i, j \leqq n,$$

or

$$\frac{\partial C}{\partial p_i} - \frac{\partial C}{\partial p_j} R^{ij}(p, Y) = 0,$$

the observable behavior of the demand system of the factor inputs may be expressed as the *invariant Lie group action of infinitesimal transformations* on the cost function, i.e., $L_{ij}C = 0$, where L_{ij} is an infinitesimal transformation equal to $(\partial/\partial p_i) - R^{ij}(p, Y)\, \partial/\partial p_j$ (the exact definition is given in Chapter 2 and the Appendix). Thus the study of observable market behavior of the demand functions of the factor inputs requires the study of the invariance property of partial differential equations. Second, to assume the *stable* relationships such as $R^{ij}(p, Y)$ implies that the forms of x_i and x_j are not affected or invariant under changes in some variables which are explicitly or implicitly observable. (One of the obvious variables is the index of technical change.) But from the mathematical point of view, the "stability" of the forms x_i and x_j is nothing but the "invariance" of differential equations under some transformations. Thus analysis of demand functions for factor inputs and of the cost and production functions requires again the study of invariant partial differential equations.