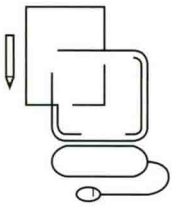

# Numerical Analysis 2000, Volume 1
# Approximation Theory

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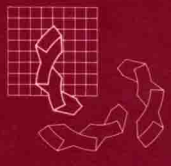

# Numerical Analysis 2000, Volume 1
# Approximation Theory

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# Approximation Theory

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Preface

# Numerical Analysis in the 20th Century Vol. I: Approximation Theory

The field of numerical analysis has witnessed many significant developments in the 20th century and will continue to enjoy major new advances in the years ahead. Therefore, it seems appropriate to compile a "state-of-the-art" volume devoted to numerical analysis in the 20th century. This volume on "Approximation Theory" is the first of seven volumes that will be published in this Journal. It brings together the papers dealing with historical developments, survey papers and papers on recent trends in selected areas.

In his paper, G.A. Watson gives an *historical survey* of methods for solving approximation problems in normed linear spaces. He considers approximation in $L_p$ and Chebyshev norms of real functions and data. Y. Nievergelt describes the history of least-squares approximation. His paper surveys the development and applications of ordinary, constrained, weighted and total least-squares approximation. D. Leviatan discusses the degree of approximation of a function in the uniform or $L_p$-norm.

The development of numerical algorithms is strongly related to the type of approximating functions that are used, e.g. orthogonal polynomials, splines and wavelets, and several authors describe these different approaches.

E. Godoy, A. Ronveaux, A. Zarzo, and I. Area treat the topic of classical *orthogonal polynomials.* R. Piessens, in his paper, illustrates the use of Chebyshev polynomials in computing integral transforms and for solving integral equations.

Some developments in the use of *splines* are described by G. Nürnberger, F. Zeilfelder (for the bivariate case), and by R.-H. Wang in the multivariate case. For the numerical treatment of functions of several variables, radial basis functions are useful tools. R. Schaback treats this topic in his paper. Certain aspects of the computation of Daubechie *wavelets* are explained and illustrated in the paper by C. Taswell. P. Guillaume and A. Huard explore the case of multivariate Padé approximation.

*Special functions* have played a crucial role in approximating the solutions of certain scientific problems. N. Temme illustrates the usefulness of parabolic cylinder functions and J.M. Borwein, D.M. Bradley, R.E. Crandall provide a compendium of evaluation methods for the Riemann zeta function. S. Lewanowicz develops recursion formulae for basic hypergeometric functions. Aspects of the spectral theory for the classical Hermite differential equation appear in the paper by W.N. Everitt, L.L. Littlejohn and R. Wellman.

Many *applications* of approximation theory are to be found in linear system theory and model reduction. The paper of B. De Schutter gives an overview of minimal state space realization in

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linear system theory and the paper by A. Bultheel and B. De Moor describes the use of rational approximation in linear systems and control.

For problems whose solutions may have singularities or infinite domains, *sinc approximation methods* are of value. F. Stenger summarizes the results in this field in his contribution.

G. Alefeld, and G. Mayer, provide a survey of the historical development of *interval analysis*, including several applications of interval mathematics to numerical computing.

These papers illustrate the profound impact that ideas of approximation theory have had in the creation of numerical algorithms for solving real-world scientific problems. Furthermore, approximation- theoretical concepts have proved to be basic tools in the analysis of the applicability of these algorithms.

We thank the authors of the above papers for their willingness to contribute to this volume. Also, we very much appreciate the referees for their role in making this volume a valuable source of information for the next millennium.

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# Approximation in normed linear spaces

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**Abstract**

A historical account is given of the development of methods for solving approximation problems set in normed linear spaces. Approximation of both real functions and real data is considered, with particular reference to $L_p$ (or $l_p$) and Chebyshev norms. As well as coverage of methods for the usual linear problems, an account is given of the development of methods for approximation by functions which are nonlinear in the free parameters, and special attention is paid to some particular nonlinear approximating families. © 2000 Elsevier Science B.V. All rights reserved.

## 1. Introduction

The purpose of this paper is to give a historical account of the development of numerical methods for a range of problems in best approximation, that is problems which involve the minimization of a norm. A treatment is given of approximation of both real functions and data. For the approximation of functions, the emphasis is on the use of the Chebyshev norm, while for data approximation, we consider a wider range of criteria, including the other $l_p$ norms, $1 \leqslant p < \infty$. As well as the usual linear problems, a general account is given of nonlinear best approximation, and we also consider some special cases. Only a passing mention is made of least-squares problems, as that is considered elsewhere. The focus is also entirely on the approximation of real quantities, and so best approximation of complex quantities is not covered. A partial justification of this is that dealing with problems in generality as complex ones would introduce additional complication not entirely justified by the additional algorithmic initiatives.

Since we are concerned here with historical development, technical details are not included for their own sake. The intention is, where appropriate, to be descriptive, rather to give a technically rigorous and detailed account of methods. However, it seemed necessary at times for the sake of comprehensibility, and in order to fully appreciate algorithmic developments, to include a reasonable amount of technical detail.

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Obviously a major factor in the development of methods has been the advent of powerful computing facilities, as this has opened up opportunities to tackle a wide range of practical problems. Whereas at one time, the main consideration may have been elegance and simplicity, with attention perhaps focussed on a set of problems satisfying "classical" assumptions, those considerations now usually have to take second place to the treatment of problems which are seen to be of practical importance, for which algorithms have to be robust and efficient.

The paper is effectively divided into two parts, the first (Section 2) being concerned with approximation by linear families, and the second (Section 3) being concerned with approximation by nonlinear families. These sections themselves further subdivide into two parts, where we consider separately approximation of data and of functions, and these are dealt with in that order within the two sections, with a further breakdown in what seems to be a reasonably natural way to take account of important special cases.

For the approximation of functions, we are primarily concerned with univariate functions on an interval $[a,b]$, because that is where most effort has been concentrated. However, some relevant comments are made on the extent to which multivariate functions may also be treated, with a few references made to this.

## 2. Linear approximation

The approximation of a given function defined on an interval by a linear combination of given functions is the most fundamental problem in approximation theory. The functions involved are usually continuous, and this can be thought of as a continuous infinite dimensional approximation problem. If the functions are replaced by vectors in $\mathbb{R}^m$, then we have a class of finite dimensional or discrete problems, many of which have their origins in data fitting. That solutions to linear best approximation problems always exist is a result which goes back at least to Riesz in 1918 [174]. We will consider the finite dimensional problem first, and begin by making some general remarks, before looking at special cases.

### 2.1. Linear approximation in $\mathbb{R}^m$

Let $A \in \mathbb{R}^{m\times n}$ where $m \geqslant n$, and let $\boldsymbol{b} \in \mathbb{R}^m$. Then the statement of a linear best approximation problem in $\mathbb{R}^m$ can be given as

$$\text{find } \boldsymbol{x} \in \mathbb{R}^n \text{ to minimize } ||\boldsymbol{r}||, \tag{1}$$

where

$$\boldsymbol{r} = A\boldsymbol{x} - \boldsymbol{b},$$

and $||.||$ is a given norm on $\mathbb{R}^m$. The dependence of $\boldsymbol{r}$ on $\boldsymbol{x}$ will generally be suppressed, unless confusion is possible.

This particular problem has attracted enormous interest. It will be assumed throughout that $\text{rank}(A) = n$, and there is no $\boldsymbol{x}$ such that $\boldsymbol{r} = 0$. These are not essential, neither in theory nor in practice; however, they are conditions that are normally satisfied in practice, and their assumption considerably simplifies the presentation. If the norm is a differentiable function of $\boldsymbol{x}$, then we can easily characterize a minimum by zero derivative conditions: these are necessary, and, exploiting

convexity, also sufficient. The best known example is when the norm is the least-squares norm, when zero derivative conditions just give the usual normal equations

$$A^{\mathrm{T}}A\boldsymbol{x} = A^{\mathrm{T}}\boldsymbol{b}.$$

The method of least squares is considered in detail elsewhere. But in a data fitting context, other $l_p$ norms, particularly those for values of $p$ satisfying $1 \leqslant p < 2$ are also important. The reason for this is that it is common for the usual conditions justifying the use of the $l_2$ norm not to hold, for example there may be wild points or gross errors in the data, and these other norms give reduced weight to these wild points. This is considered in Sections 2.2 and 2.3. Of great interest also has been the use of the Chebyshev norm; this is perhaps of less value in a data fitting context, but problems arise for example in continuous function approximation when the region of approximation is discretized. The problem is rich in structure and the theory is a beautiful one; we consider this case in Section 2.4.

We will restrict attention here to the problem (1), although there are many modifications of that problem which are relevant in a data fitting context. Most modifications have only been given serious treatment comparatively recently, and so they are of lesser interest from a historical point of view.

### 2.2. Linear $l_1$ approximation in $\mathbb{R}^m$

Consider now the problem (1) with the $l_1$ norm

$$||\boldsymbol{r}||_1 = \sum_{i=1}^{m} |r_i|. \tag{2}$$

This problem has a long history: its statement goes back well into the mid eighteenth century, and predates the introduction of least squares. Certainly, it was used in work of Laplace in 1786, in solving the overdetermined system of linear equations determining planetary orbits [110]. The first systematic methods for solving this problem seem due to Edgeworth [61]; in 1887 he gave a method based on tabulation, and in 1888 a method for the case when $n = 2$ which was essentially graphical and conceptual, but based on calculating descent directions. In 1930, Rhodes [167], motivated by the problem of fitting a parabola to data, tried Edgeworth's later method but found it "cumbrous". He gave a method where each iteration was calculated by solving 2 interpolation conditions for 2 of the parameters, and minimizing with respect to the remaining parameter. A proof that this kind of approach can give a solution was established by Singleton in 1940 [182]. A detailed historical account is given by Farebrother in a 1987 paper [63], covering the period 1793 to 1930.[1]

The first modern systematic study of this problem appears to be by Motzkin and Walsh [131,132] in the late 1950s, and characterization results are given in the 1964 book by Rice [172]. A convenient form of these may be deduced from these results or as a simple consequence of applying to this special case known results in abstract approximation theory: we will not attempt to go down that historical route, since it is something of a diversion from the main theme. However, it is the case

[1] The 1999 book by Farebrother [64] is also relevant.

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that a vector $\boldsymbol{x} \in \mathbb{R}^n$ solves the $l_1$ problem if and only if there exists a vector $\boldsymbol{v} \in \mathbb{R}^m$ satisfying

$$A^{\mathrm{T}}\boldsymbol{v} = 0,$$

where $||\boldsymbol{v}||_\infty \leqslant 1$, and $v_i = \operatorname{sign}(r_i)$ whenever $r_i \neq 0$. The first simple (direct) proof of this was probably given by Watson [199] in 1980. A formal treatment of the important result that when $A$ has rank $n$, a solution will be such that $n$ components of $r_i$ are zero, was given by Motzkin and Walsh [131] in 1955. In the context of the $l_1$ problem, any point characterized in this way can be defined to be a *vertex*. The interpolation result (in special cases) appears to have been known to Gauss, and to have been used in early methods: for example, the methods of Rhodes and Singleton are essentially vertex to vertex descent methods.

The results of Motzkin and Walsh were arrived at by direct consideration of the problem. However, its relationship with a linear programming problem was recognized around the same time,[2] and linear programming theory provides a parallel route to the same properties. Around 1947, Dantzig did his pioneering work on the simplex method of linear programming, and over the next few years, duality theory was developed, largely by von Neumann, Gale, Kuhn and Tucker. The significance of these developments for numerical methods for the $l_1$ (and the $l_\infty$) problem cannot be overemphasized.

The first representation of the $l_1$ problem as a tractable linear programming problem seems due to Charnes et al. [35] in 1955. The key observation is that if extra variables $\boldsymbol{u}$ and $\boldsymbol{v} \in \mathbb{R}^m$ are introduced, then the problem can be posed as

$$\text{minimize } \sum_{i=1}^{m} (u_i + v_i) \text{ subject to} \tag{3}$$

$$[I : -I : A] \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{x} \end{bmatrix} = \boldsymbol{b}$$

$$\boldsymbol{u} \geqslant 0, \quad \boldsymbol{v} \geqslant 0.$$

Since in the simplex method, no columns of $I$ and $-I$ can simultaneously be basic, then

$$u_i v_i = 0, \quad i = 1, \ldots, m.$$

It follows that $u_i + v_i = |u_i - v_i|$ for all $i$ and the equivalence of the simplex method applied to this problem with the minimization of (2) can readily be established.

Another version of the primal can be stated:

$$\text{minimize } \boldsymbol{e}^{\mathrm{T}}\boldsymbol{s} \text{ subject to}$$

$$-\boldsymbol{s} \leqslant A\boldsymbol{x} - \boldsymbol{b} \leqslant \boldsymbol{s}.$$

This goes back at least to the 1964 Russian edition of the book by Zuhovitskii and Avdeyeva [211]. However, this form of the problem does not seem to have attracted as much attention as (3). The zero residuals will result in a form of degeneracy.

---

[2] Farebrother [63] in his 1987 paper interprets the work of Edgeworth in this context, and states that "..it must be conceded that Edgeworth had developed a fully operational, if somewhat complex, linear programming procedure for the $L_1$ estimation problem in 1888".

Fisher [66] in 1961 gave some publicity to (3) for the benefit of the statistical community, and this form was also used by Barrodale and Young [13] in 1966, who provided an Algol implementation and numerical results. The fact that the components of $\boldsymbol{x}$ may be non-negative is not a major problem in this context: for example, they can each be replaced by the difference of two non-negative variables. It was also noted that no first phase simplex calculation is required because an initial basic feasible solution can readily be obtained: if $b_i < 0$ then $\boldsymbol{e}_i$ can be present in the initial basis, if $b_i > 0$ then $-\boldsymbol{e}_i$ can be, with either used if $b_i = 0$.

The linear programming connection is sometimes wrongly credited to Wagner [192] in 1959, who posed the problem as a bounded variable or interval programming problem. In fact the form of the problem considered by Wagner [192] can be interpreted as the dual of (3). This can be written as

$$\text{maximize } \boldsymbol{b}^{\mathrm{T}}\boldsymbol{v} \text{ subject to} \tag{4}$$

$$A^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{0}$$

$$-\boldsymbol{e} \leqslant \boldsymbol{v} \leqslant \boldsymbol{e},$$

where $\boldsymbol{e}$ is a vector with every component equal to 1. Attention was re-focussed on (4) by Robers and Ben-Israel [175] in 1969, and Robers and Robers [176] in 1973, who argued the advantages of that approach, which included computational efficiency: the problem with the primal appeared to be the large number of extra variables required. However, an improved version of the primal linear programming method was given by Davies [53] in 1967 and Barrodale and Roberts [10] in 1970, where a special pivot column selection rule was employed, and in 1973, both Spyropoulos et al. [183] and Barrodale and Roberts [11] gave efficient implementations of the simplex method applied to the primal which fully exploited the structure. The Barrodale and Roberts method achieved efficiency by taking multiple pivot steps, exploiting the fact that descent can continue beyond the usual point when feasibility is lost, because feasibility can readily be recovered by swapping certain variables into and out of the basis. Further efficiency was achieved by imposing certain restrictions on the choice of variables to enter and leave the basis. A Fortran programme and numerical results were provided, together with favourable comparisons with some other primal and dual methods [12].

In 1975, Abdelmalik [2] developed a special purpose method for the dual, using the dual simplex method, and his method seemed comparable with that of Barrodale and Roberts [10]. This turned out not really to be surprising, because, as pointed out by Armstrong and Godfrey [6] in 1979, the application of the dual simplex method to the dual is equivalent to applying the primal simplex method to the primal. So apart from implementation aspects, the methods were the same.

A basic feasible solution to (3) in which all columns of $A$ are present in the basis can readily be shown to correspond to a vertex as defined above. Therefore, once the columns of $A$ are present in the basis, the simplex method is a vertex to vertex descent method. There are many other variants of these linear programming methods, but away from a linear programming context, *direct* descent methods were being considered. For given $\boldsymbol{x}$, let

$$Z = \{i : r_i = 0\}.$$

Then since for full rank problems the solution occurs at a point $\boldsymbol{x}$ with $Z$ containing $n$ indices (a vertex), we want to systematically descend to such a point. Perhaps the first modern direct descent methods were given by Usow [189] in 1967, and Claerbout and Muir [43] in 1973. A natural way

to implement descent methods is by first finding a vertex, and then descending through a sequence of vertices. Thus there are two types of step depending on whether at the current point, $Z$ contains (a) fewer than $n$ indices (b) exactly $n$ indices. (The possibility that $Z$ contains more than $n$ indices corresponds to a degenerate situation, and although there are ways round it, will for our purposes be ignored.) Then in case (a) movement as far as possible is made in the direction $\boldsymbol{d}$ in such a way that the number of indices in $Z$ at the new point is increased, and in case (b) movement as far as possible is made in the direction $\boldsymbol{d}$ in such a way that the number of indices in $Z$ is maintained. Effective methods of this type, therefore, have this strategy in common, and are distinguished by the way the descent direction is calculated. There are mainly two approaches, (i) *reduced gradient methods*, where the "active constraints" are used to express certain variables in terms of others, the objective function is expressed in terms of the latter group, and its gradient is obtained in terms of those, and (ii) *projected gradient methods*, where the gradient is obtained by projecting the gradient of the objective function onto the orthogonal complement of the span of the active constraints.

Bartels et al. [18] in 1978 gave a projected gradient method, and reduced gradient methods were given by Osborne [147,148] in 1985 and 1987. Both projected and reduced gradient methods were analyzed in detail by Osborne [147] in 1985, and he pointed out that although reduced gradient methods seem more suitable for implementation using a tableau format, with updating, in fact such organization is available for implementing both methods. On relationships with linear programming methods, he showed that there is an exact equivalence between the possible options available in implementing the simplex method and those available in the direct application of the reduced gradient method. Thus these algorithms are equivalent: only the implementational details are different. The usual simplex step corresponds to a particular option in the reduced gradient method, based on an unnormalized steepest edge test for determining the variable to leave the basis. A different way of choosing this variable (a normalized steepest edge test, which is scale invariant) was used by Bloomfield and Stieger [26] in 1983, and their evidence showed that this can lead to improvement.

Nearly all the good methods considered to the end of the 1980s were vertex to vertex methods, which exploit the polyhedral nature of the function to be minimized, and (in the absence of degeneracy) they are finite. There has been recent interest in interior point methods for linear programming problems, stimulated by the results of Karmarker [102] in 1984. In conjunction with a formal connection with classical barrier methods for constrained optimization problems, this has resulted in renewed interest in linear programming, and there has of course been an impact on special cases such as the $l_1$ problem.

The use of interior point methods for $l_1$ problems goes back at least as far as work of Meketon [127] in 1987, and methods have been given since then by Ruzinsky and Olsen [178] in 1989, Zhang in 1993 [209] and Duarte and Vanderbei [56] in 1994. Portnoy and Koenker [157] in 1997 make a case for the superiority of interior point methods over simplex-based methods for large problems. Based on comparisons of $l_1$ problems having $n$ up to 16 and $m$ from 1000 to 200 000, they conclude that there is "a compelling general case for the superiority of interior point methods over traditional simplex methods for large linear programming problems". Their algorithm of choice for the $l_1$ problem is based on a primal–dual log barrier method due to Mehrotra [123] in 1992, and includes a statistical preprocessing approach which estimates whether a residual is zero or not. The opposition is represented by a variant of the Barrodale and Roberts method.

Meantime, two other types of smoothing method were being developed for the $l_1$ problem.[3] The first of these is typified by an algorithm of Coleman and Li [46] in 1992, which is based on affine scaling: while not strictly speaking an interior point method, it is nevertheless in the spirit of such methods. Here, an attempt is made to satisfy the characterization conditions by an iterative descent method which has the following characteristics: (a) it generates a sequence of points which are such that $Z$ is empty, so that derivatives exist, (b) it is globally convergent, (c) it ultimately takes damped Newton steps (damped to satisfy (a)), but with sufficiently accurate approximations to the full Newton step to permit quadratic convergence (under nondegeneracy conditions). Careful implementation of the method can avoid difficulties with near-zero components of $\boldsymbol{r}$ and the approach seems promising for large problems as it is insensitive to problem size. Some comparisons show that it is superior to Meketon's interior point method for problems with $n$ up to 200, $m$ up to 1000.

A second approach to smoothing the $l_1$ problem was developed by Madsen and Nielsen [116] in 1993. It is based on the use of the Huber M-estimator, defined by

$$\psi_\gamma \equiv \psi_\gamma(\boldsymbol{r}) = \sum_{i=1}^{m} \rho(r_i), \tag{5}$$

where

$$\rho(t) = \begin{cases} t^2/2, & |t| \leqslant \gamma, \\ \gamma(|t| - \gamma/2), & |t| > \gamma, \end{cases} \tag{6}$$

and $\gamma$ is a scale factor or tuning constant. The function (5) is convex and once continuously differentiable, but has discontinuous second derivatives at points where $|r_i| = \gamma$. The mathematical structure of the Huber M-estimator seems first to have been considered in detail by Clark [44] in 1985. Clearly if $\gamma$ is chosen large enough, then $\psi_\gamma$ is just the least-squares function; in addition if $\gamma$ tends to zero, then limit points of the set of solutions may be shown to minimize the $l_1$ norm. It is the latter property which concerns us here.

It has been suggested by Madsen and Nielsen [116] in 1993 and also by Li and Swetits [113] in 1998 that the preferred method for solving the $l_1$ problem is via a sequence of Huber problems for a sequence of scale values $\gamma \to 0$. This algorithmic development has lead to increased interest in the relationship between the Huber M-estimator and the $l_1$ problem; for example there is recent work of Madsen et al. [117] in 1994, and Li and Swetits [113] in 1998. The method of Madsen and Nielsen generates Huber solutions for a sequence of values of $\gamma$, tending to zero. The solutions are obtained by solving least-square problems, exploiting structure so that new solutions can be obtained using updating often in $O(n^2)$ operations. A key feature is that it is not necessary to let $\gamma$ reach zero; once a sufficiently small value is identified, then the $l_1$ solution may be obtained by solving an $n \times n$ linear system. Madsen and Neilsen give some comparisons (for randomly generated problems, and with $m$ mostly set to $2n$ for $m$ up to 1620) with the method of Barrodale and Roberts [10] and claim superiority.

An important issue as far as the implementation of simplex type methods is the efficiency of the line search. The Barrodale and Roberts [10] method incorporates the equivalent of a comparison sort, and this leaves room for considerable improvement. Bloomfield and Stieger [26] considered

[3] The observation that a best approximation can always be computed as the limit of a sequence of $l_p$ approximations as $p \to 1$ is due to Fischer [65] in 1983 (an algorithm based on this was in fact given by Abdelmalik [1] in 1971), although this is not a very practical approach.

this aspect in their 1983 book, and suggested using a fast median method. An alternative based on the use of the secant algorithm was considered (in a related context) by George and Osborne [71] in 1990, and again by Osborne [147] in 1985. Numerical experiments were reported by Osborne and Watson [154] in 1996, where the secant-based method was seen to be as good as fast median methods on randomly generated problems, and to perform considerably better on problems with systematic data. Comparisons of other types of method with simplex methods really need to take this into account before definitive conclusions can be drawn.

### 2.3. Linear $l_p$ approximation in $\mathbb{R}^m$, $1 < p < \infty$, $p \neq 2$

For given $\boldsymbol{x} \in \mathbb{R}^n$, let $D_{|r|}$ be defined by

$$D_{|\boldsymbol{r}|} = \operatorname{diag}\{|r_1|, \ldots, |r_m|\}.$$

Then $\boldsymbol{x}$ minimizes

$$||\boldsymbol{r}||_p^p = \sum_{i=1}^{m} |r_i|^p$$

with $1 < p < \infty$ if and only if derivatives with respect to $\boldsymbol{x}$ are zero, that is if

$$A^{\mathrm{T}} D_{|r|}^{p-1} \theta = 0, \tag{7}$$

where $\theta_i = \operatorname{sign}(r_i)$, $i = 1, \ldots, m$. This is a nonlinear system of equations for $\boldsymbol{x}$.

This criterion (for $p$ even) was mentioned by Gauss as a generalization of his least-squares criterion. Apart from this special case, the more general $l_p$ problem only seems to have attracted relatively recent computational attention. The range $1 < p < 2$ is of particular interest computationally because there is potentially reduced smoothness: problems with $p \geqslant 2$ are twice differentiable, but problems with $1 < p < 2$ may be only once differentiable. If $p \geqslant 2$ or if $1 < p < 2$ and no component of $\boldsymbol{r}$ is zero then twice differentiability is guaranteed and so (7) can be written as

$$A^{\mathrm{T}} D \boldsymbol{r} = 0, \tag{8}$$

where

$$D = D_{|\boldsymbol{r}|}^{p-2},$$

and this is a particularly convenient form with which to work. It represents a generalized system of normal equations, effectively a least-squares problem apart from the "weighting" matrix $D$. Fixing $\boldsymbol{x}$ to an approximate value in $D$ and solving this weighted system for a new approximation gives an example of the technique known as *iteratively reweighted least squares* or IRLS, which seems to have been introduced by Beaton and Tukey [20] in 1974. Since good software for (weighted) least-squares problems was then available, this seemed an attractive idea, additionally so since there are some apparently good theoretical properties: this simple iteration process will converge locally if $p$ is close to 2, and if zero components of $\boldsymbol{r}$ are avoided, it is globally convergent (from any initial approximation) for $1 < p < 2$. The last result seems first to have been given by Dutter [60] in 1975. However, convergence can be slow, particularly as $p$ nears 1 (it is linear with convergence constant $|p-2|$, as shown by Wolfe [206] in 1979), and there are potential numerical difficulties for reasons which will be clear from the previous section. The matrix $D$ (which may not exist) can

be replaced by approximations (even by the unit matrix), and this gives rise to variants of the IRLS technique, but again convergence can be very slow.

Most recent algorithms for solving (8) are based on Newton's method, and many variants were proposed in the 1970s. It is interesting that the Newton step is just $1/(p-1)$ times the IRLS step (as measured by the difference between successive approximations), as pointed out by Watson [196] in 1977, and this gave an explanation of some success obtained by Merle and Späth [128] in 1974 in using a damped IRLS procedure with step length $(p-1)$. Thus apart from differences due to the line search, IRLS and Newton's method with line searches *are essentially the same method.* It is easily seen that the region of convergence of Newton's method is proportional to $|(p-1)/(p-2)|$, so good line search procedures are needed even with the basic method, certainly far from $p=2$. However, for $p>2$, Newton's method with line search is usually perfectly satisfactory.

Since from a practical point of views the interesting cases are those when $1<p<2$, different strategies have been proposed for getting round the difficulties arising from zero (or near zero) components of $\boldsymbol{r}$. These included the substitution of small nonzero values, solving a slightly perturbed problem, or identifying and so removing these components from the set. However, not just zero components but *nearly zero* components are potentially troublesome. There is some evidence, however, that these phenomena are not *by themselves* a major problem, but only if they are accompanied by $p$ being close to 1. The main difficulty appears to be due the fact that as $p$ approaches 1, we are coming closer to a discontinuous problem, effectively to a constrained problem. It seems necessary to recognize this in a satisfactory algorithm, and consider some of the elements of the $l_1$ problem in devising an approach which will deal in a satisfactory with small values of $p$. This is the philosophy in a recent method due to Li [114] in 1993, which is essentially equivalent to the method for the $l_1$ problem of Coleman and Li [46] referred to in the previous section. Numerical results show that the new method is clearly superior to IRLS (with the same line search) for values of $p$ close to 1, with the gap between the two methods widening as $p$ approaches 1. There is little difference for values of $p \geqslant 1.5$ or so. As with the $l_1$ case, the number of iterations appears to be independent of the problem size.

### 2.4. Linear Chebyshev approximation in $\mathbb{R}^m$

The use of the criterion now known as the Chebyshev norm

$$||\boldsymbol{r}||_\infty = \max_i |r_i|, \tag{9}$$

seems to go back to Laplace in 1786, who gave a solution procedure for $n=2$. Cauchy in 1814 and Fourier in 1824 gave descent methods. A detailed historical account is given by Farebrother [63] in his 1987 paper, covering the period 1793 to 1824. The function space analogue was studied first by Chebyshev[4] from the 1850s, arising from an analysis of a steam engine linking, and both continuous and discrete problems now carry his name.

For any $\boldsymbol{x} \in \mathbb{R}^n$, let

$$\bar{I}(\boldsymbol{x}) = \{i\colon |r_i(\boldsymbol{x})| = ||\boldsymbol{r}||_\infty\}.$$

[4] The number of variants in the western literature which have been used for Chebyshev is legendary, but most people now seemed to have settled on this one. Butzer and Jongmans [33] in 1999 gave a detailed account of Chebyshev's life and work.

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