

# Dualities and Representations of Lie Superalgebras

**Shun-Jen Cheng**  
**Weiqliang Wang**

**Graduate Studies  
in Mathematics**

**Volume 144**



**American Mathematical Society**

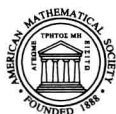
# Dualities and Representations of Lie Superalgebras

Shun-Jen Cheng  
Weiqiang Wang



Graduate Studies  
in Mathematics

Volume 144



American Mathematical Society  
Providence, Rhode Island

## EDITORIAL COMMITTEE

David Cox (Chair)  
Daniel S. Freed  
Rafe Mazzeo  
Gigliola Staffilani

2010 *Mathematics Subject Classification*. Primary 17B10, 17B20.

---

For additional information and updates on this book, visit  
**[www.ams.org/bookpages/gsm-144](http://www.ams.org/bookpages/gsm-144)**

---

### Library of Congress Cataloging-in-Publication Data

Cheng, Shun-Jen, 1963–

Dualities and representations of Lie superalgebras / Shun-Jen Cheng, Weiqiang Wang.  
pages cm. — (Graduate studies in mathematics ; volume 144)

Includes bibliographical references and index.

ISBN 978-0-8218-9118-6 (alk. paper)

1. Lie superalgebras. 2. Duality theory (Mathematics) I. Wang, Weiqiang, 1970– II. Title.

QA252.3.C44 2013  
512'.482—dc23

2012031989

---

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294 USA. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2012 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1      17 16 15 14 13 12

# Dualities and Representations of Lie Superalgebras

**To Mei-Hui, Xiaohui, Isabelle, and our parents**

---

# Preface

Lie algebras, Lie groups, and their representation theories are parts of the mathematical language describing symmetries, and they have played a central role in modern mathematics. An early motivation of studying Lie superalgebras as a generalization of Lie algebras came from supersymmetry in mathematical physics. Ever since a Cartan-Killing type classification of finite-dimensional complex Lie superalgebras was obtained by Kac [60] in 1977, the theory of Lie superalgebras has established itself as a prominent subject in modern mathematics. An independent classification of the finite-dimensional complex simple Lie superalgebras whose even subalgebras are reductive (called simple Lie superalgebras of classical type) was given by Scheunert, Nahm, and Rittenberg in [106].

The goal of this book is a systematic account of the structure and representation theory of finite-dimensional complex Lie superalgebras of classical type. The book intends to serve as a rigorous introduction to representation theory of Lie superalgebras on one hand, and, on the other hand, it covers a new approach developed in the past few years toward understanding the Bernstein-Gelfand-Gelfand (BGG) category for classical Lie superalgebras. In spite of much interest in representations of Lie superalgebras stimulated by mathematical physics, these basic topics have not been treated in depth in book form before. The reason seems to be that the representation theory of Lie superalgebras is dramatically different from that of complex semisimple Lie algebras, and a systematic, yet accessible, approach toward the basic problem of finding irreducible characters for Lie superalgebras was not available in a great generality until very recently.

We are aware that there is an enormous literature with numerous partial results for Lie superalgebras, and it is not our intention to make this book an encyclopedia. Rather, we treat in depth the representation theory of the three most important classes of Lie superalgebras, namely, the general linear Lie superalgebras  $\mathfrak{gl}(m|n)$ ,

the ortho-symplectic Lie superalgebras  $\mathfrak{osp}(m|2n)$ , and the queer Lie superalgebras  $\mathfrak{q}(n)$ . To a large extent, representations of  $\mathfrak{sl}(m|n)$  can be understood via  $\mathfrak{gl}(m|n)$ . The lecture notes [32] by the authors can be considered as a prototype for this book. The presentation in this book is organized around three dualities with a unifying theme of determining irreducible characters:

Schur duality,   Howe duality,   and   super duality.

The new book of Musson [90] treats in detail the ring theoretical aspects of the universal enveloping algebras of Lie superalgebras as well as the basic structures of simple Lie superalgebras.

There are two superalgebra generalizations of Schur duality. The first one, due to Sergeev [110] and independently Berele-Regev [7], is an interplay between the general linear Lie superalgebra  $\mathfrak{gl}(m|n)$  and the symmetric group, which incorporates the trivial and sign modules in a unified framework. The irreducible polynomial characters of  $\mathfrak{gl}(m|n)$  arising this way are given by the super Schur polynomials. The second one, called Sergeev duality, is an interplay between the queer Lie superalgebra  $\mathfrak{q}(n)$  and a twisted hyperoctahedral group algebra. The Schur  $Q$ -functions and related combinatorics of shifted tableaux appear naturally in the description of the irreducible polynomial characters of  $\mathfrak{q}(n)$ .

It has been observed that much of the classical invariant theory for the polynomial algebra has a parallel theory for the exterior algebra as well. The First Fundamental Theorem (FFT) for both polynomial invariants and skew invariants for classical groups admits natural reformulation and extension in the theory of Howe's reductive dual pairs [51, 52]. Lie superalgebras allow an elegant and uniform treatment of Howe duality on the polynomial and exterior algebras (cf. Cheng-Wang [29]). For the general linear Lie groups, Schur duality, Howe duality, and FFT are equivalent. Unlike Schur duality, Howe duality treats classical Lie groups and (super)algebras beyond type  $A$  equally well. The Howe dualities allow us to determine the character formulas for the irreducible modules appearing in the dualities.

The third duality, super duality, has a completely different flavor. It views the representation theories of Lie superalgebras and Lie algebras as two sides of the same coin, and it is an unexpected and rather powerful approach developed in the past few years by the authors and their collaborators, culminating in Cheng-Lam-Wang [24]. The super duality approach allows one to overcome in a conceptual way various major obstacles in super representation theory via an equivalence of module categories of Lie algebras and Lie superalgebras.

Schur, Howe, and super dualities provide approaches to the irreducible character problem in increasing generality and sophistication. Schur and Howe dualities only offer a solution to the irreducible character problem for modules in some semisimple subcategories. On the other hand, super duality provides a conceptual solution to the long-standing irreducible character problem in fairly general BGG

categories (including all finite-dimensional modules) over classical Lie superalgebras in terms of the usual Kazhdan-Lusztig polynomials of classical Lie algebras. Totally different and independent approaches to the irreducible character problem of *finite-dimensional*  $\mathfrak{gl}(m|n)$ -modules have been developed by Serganova [107] and Brundan [11]. Also Brundan's conjecture on irreducible characters of  $\mathfrak{gl}(m|n)$  in the full BGG category  $\mathcal{O}$  has recently been proved in [26]. Super duality again plays a crucial role in the proof. However, this latest approach to the full BGG category  $\mathcal{O}$  is beyond the scope of this book.

The book is largely self-contained and should be accessible to graduate students and non-experts as well. Besides assuming basic knowledge of entry-level graduate algebra (and some familiarity with basic homological algebra in the final Chapter 6), the other prerequisite is a one-semester course in the theory of finite-dimensional semisimple Lie algebras. For example, either the book by Humphreys or the first half of the book by Carter on semisimple Lie algebras is sufficient. Some familiarity with symmetric functions and representations of symmetric groups can be sometimes useful, and Appendix A provides a quick summary for our purpose. It is possible that super experts may also benefit from the book, as several "folklore" results are rigorously proved and occasionally corrected in great detail here, sometimes with new proofs. The proofs of some of these results can be at times rather difficult to trace or read in the literature (and not merely because they might be in a different language).

Here is a broad outline of the book chapter by chapter. Each chapter ends with exercises and historical notes. Though we have tried to attribute the main results accurately and fairly, we apologize beforehand for any unintended omissions and mistakes.

Chapter 1 starts by defining various classes of Lie superalgebras. For the basic Lie superalgebras, we introduce the invariant bilinear forms, root systems, fundamental systems, and Weyl groups. Positive systems and fundamental systems for basic Lie superalgebras are not conjugate under the Weyl group, and the notion of odd reflections is introduced to relate non-conjugate positive systems. The PBW theorem for the universal enveloping algebra of a Lie superalgebra is formulated, and highest weight theory for basic Lie superalgebras and  $\mathfrak{q}(n)$  is developed.

In Chapter 2, we focus on Lie superalgebras of types  $\mathfrak{gl}$ ,  $\mathfrak{osp}$  and  $\mathfrak{q}$ . We classify their finite-dimensional simple modules using odd reflection techniques. We then formulate and establish precisely the images of the respective Harish-Chandra homomorphisms and linkage principles. We end with a Young diagrammatic description of the extremal weights in the simple polynomial  $\mathfrak{gl}(m|n)$ -modules and finite-dimensional simple  $\mathfrak{osp}(m|2n)$ -modules. It takes considerably more effort to formulate and prove these results for Lie superalgebras than for semisimple Lie algebras because of the existence of non-conjugate Borel subalgebras and the limited role of Weyl groups for Lie superalgebras.

Schur duality for Lie superalgebras is developed in Chapter 3. We start with some results on the structure of associative superalgebras including the super variants of the Wedderburn theorem, Schur's lemma, and the double centralizer property. The Schur-Sergeev duality for  $\mathfrak{gl}(m|n)$  is proved, and it provides a classification of irreducible polynomial  $\mathfrak{gl}(m|n)$ -modules. As a consequence, the characters of the simple polynomial  $\mathfrak{gl}(m|n)$ -modules are given by the super Schur polynomials. On the algebraic combinatorial level, there is a natural super generalization of the notion of semistandard tableau, which is a hybrid of the traditional version and its conjugate counterpart. The Schur-Sergeev duality for  $\mathfrak{q}(n)$  requires understanding the representation theory of a twisted hyperoctahedral group algebra, which we develop from scratch. The characters of the simple polynomial  $\mathfrak{q}(n)$ -modules are given by the Schur  $Q$ -polynomials up to some 2-powers.

In Chapter 4, we give a quick introduction to classical invariant theory, which serves as a preparation for Howe duality in the next chapter. We describe several versions of the FFT for the classical groups, i.e., a tensor algebra version, a polynomial algebra version, and a supersymmetric algebra version.

Howe duality is the main topic of Chapter 5. Like Schur duality, Howe duality involves commuting actions of a classical Lie group  $G$  and a classical superalgebra  $\mathfrak{g}'$  on a supersymmetric algebra. The precise relation between the classical Lie superalgebras and Weyl-Clifford algebras  $\mathcal{WC}$  is established. According to the FFT for classical invariant theory in Chapter 4 when applied to the  $G$ -action on the associated graded algebra  $\text{gr } \mathcal{WC}$ , the basic invariants generating  $(\text{gr } \mathcal{WC})^G$  turn out to form the associated graded space for a Lie superalgebra  $\mathfrak{g}'$ . From this it follows that the algebra of  $G$ -invariants  $\mathcal{WC}^G$  is generated by  $\mathfrak{g}'$ . Multiplicity-free decompositions for various  $(G, \mathfrak{g}')$ -Howe dualities are obtained explicitly. Character formulas for the irreducible  $\mathfrak{g}'$ -modules appearing in  $(G, \mathfrak{g}')$ -Howe duality are then obtained via a comparison with Howe duality involving classical groups  $G$  and infinite-dimensional Lie algebras, which we develop in detail.

Finally in Chapter 6, we develop a super duality approach to obtain a complete and conceptual solution of the irreducible character problem in certain parabolic Bernstein-Gelfand-Gelfand categories for general linear and ortho-symplectic Lie superalgebras. This chapter is technically more sophisticated than the earlier chapters. Super duality is an equivalence of categories between parabolic categories for Lie superalgebras and their Lie algebra counterparts at an infinite-rank limit, and it matches the corresponding parabolic Verma modules, irreducible modules, Kostant  $u$ -homology groups, and Kazhdan-Lusztig-Vogan polynomials. Truncation functors are introduced to relate the BGG categories for infinite-rank and finite-rank Lie superalgebras. In this way, we obtain a solution à la Kazhdan-Lusztig of the irreducible character problem in the corresponding parabolic BGG categories for finite-dimensional basic Lie superalgebras.

There is an appendix in the book. In Appendix A, we have included a fairly self-contained treatment of some elementary aspects of symmetric function theory, including Schur functions, supersymmetric functions and Schur  $Q$ -functions. The celebrated boson-fermion correspondence serves as a prominent example relating superalgebras to mathematical physics and algebraic combinatorics. The Fock space therein is used in setting up the Howe duality for infinite-dimensional Lie algebras in Chapter 5.

For a one-semester introductory course on Lie superalgebras, we recommend two plausible ways of using this book. A first approach uses Chapters 1, 2, 3, with possible supplements from Chapter 5 and Appendix A. A second approach uses Chapters 1, 3, 5 with possible supplements from Chapter 4 and Appendix A. It is also possible to use this book for a course on the interaction between representations of Lie superalgebras and algebraic combinatorics. The more advanced Chapter 6 can be used in a research seminar.

**Acknowledgment.** The book project started with the lecture notes [32] of the authors, which were an expanded written account of a series of lectures delivered by the second-named author in the summer school at East China Normal University, Shanghai, in July 2009. In a graduate course at the University of Virginia in Spring 2010, the second-named author lectured on what became a large portion of Chapters 3, 4, and 5 of the book. The materials in Chapter 3 on Schur duality have been used by the second-named author in the winter school in Taipei in December 2010. Part of the materials in the first three chapters have also been used by the first-named author in a lecture series in Shanghai in March 2011, and then in a lecture series by both authors in a workshop in Tehran in May 2011. We thank the participants in all these occasions for their helpful suggestions and feedback, and we especially thank Constance Baltera, Jae-Hoon Kwon, Li Luo, Jinkui Wan, and Youjie Wang for their corrections. We are grateful to Ngau Lam for his collaboration which has changed our way of thinking about the subject of Lie superalgebras.

The first-named author gratefully acknowledges the support from the National Science Council, Taiwan, and the second-named author gratefully acknowledges the continuing support of the National Science Foundation, USA.

Shun-Jen Cheng

Weiqiang Wang

---

# Contents

Preface	xiii
Chapter 1. Lie superalgebra ABC	1
§1.1. Lie superalgebras: Definitions and examples	1
1.1.1. Basic definitions	2
1.1.2. The general and special linear Lie superalgebras	4
1.1.3. The ortho-symplectic Lie superalgebras	6
1.1.4. The queer Lie superalgebras	8
1.1.5. The periplectic and exceptional Lie superalgebras	9
1.1.6. The Cartan series	10
1.1.7. The classification theorem	12
§1.2. Structures of classical Lie superalgebras	13
1.2.1. A basic structure theorem	13
1.2.2. Invariant bilinear forms for $\mathfrak{gl}$ and $\mathfrak{osp}$	16
1.2.3. Root system and Weyl group for $\mathfrak{gl}(m n)$	16
1.2.4. Root system and Weyl group for $\mathfrak{spo}(2m 2n+1)$	17
1.2.5. Root system and Weyl group for $\mathfrak{spo}(2m 2n)$	17
1.2.6. Root system and odd invariant form for $\mathfrak{q}(n)$	18
§1.3. Non-conjugate positive systems and odd reflections	19
1.3.1. Positive systems and fundamental systems	19
1.3.2. Positive and fundamental systems for $\mathfrak{gl}(m n)$	21
1.3.3. Positive and fundamental systems for $\mathfrak{spo}(2m 2n+1)$	22
1.3.4. Positive and fundamental systems for $\mathfrak{spo}(2m 2n)$	23
1.3.5. Conjugacy classes of fundamental systems	25
§1.4. Odd and real reflections	26
1.4.1. A fundamental lemma	26

1.4.2.	Odd reflections	27
1.4.3.	Real reflections	28
1.4.4.	Reflections and fundamental systems	28
1.4.5.	Examples	30
§1.5.	Highest weight theory	31
1.5.1.	The Poincaré-Birkhoff-Witt (PBW) Theorem	31
1.5.2.	Representations of solvable Lie superalgebras	32
1.5.3.	Highest weight theory for basic Lie superalgebras	33
1.5.4.	Highest weight theory for $q(n)$	35
§1.6.	Exercises	37
	Notes	40
Chapter 2.	Finite-dimensional modules	43
§2.1.	Classification of finite-dimensional simple modules	43
2.1.1.	Finite-dimensional simple modules of $gl(m n)$	43
2.1.2.	Finite-dimensional simple modules of $spo(2m 2)$	45
2.1.3.	A virtual character formula	45
2.1.4.	Finite-dimensional simple modules of $spo(2m 2n+1)$	47
2.1.5.	Finite-dimensional simple modules of $spo(2m 2n)$	50
2.1.6.	Finite-dimensional simple modules of $q(n)$	53
§2.2.	Harish-Chandra homomorphism and linkage	55
2.2.1.	Supersymmetrization	55
2.2.2.	Central characters	56
2.2.3.	Harish-Chandra homomorphism for basic Lie superalgebras	57
2.2.4.	Invariant polynomials for $gl$ and $osp$	59
2.2.5.	Image of Harish-Chandra homomorphism for $gl$ and $osp$	62
2.2.6.	Linkage for $gl$ and $osp$	65
2.2.7.	Typical finite-dimensional irreducible characters	68
§2.3.	Harish-Chandra homomorphism and linkage for $q(n)$	69
2.3.1.	Central characters for $q(n)$	70
2.3.2.	Harish-Chandra homomorphism for $q(n)$	70
2.3.3.	Linkage for $q(n)$	74
2.3.4.	Typical finite-dimensional characters of $q(n)$	76
§2.4.	Extremal weights of finite-dimensional simple modules	77
2.4.1.	Extremal weights for $gl(m n)$	77
2.4.2.	Extremal weights for $spo(2m 2n+1)$	80
2.4.3.	Extremal weights for $spo(2m 2n)$	82
§2.5.	Exercises	85
	Notes	89
Chapter 3.	Schur duality	91

§3.1. Generalities for associative superalgebras	91
3.1.1. Classification of simple superalgebras	92
3.1.2. Wedderburn Theorem and Schur's Lemma	94
3.1.3. Double centralizer property for superalgebras	95
3.1.4. Split conjugacy classes in a finite supergroup	96
§3.2. Schur-Sergeev duality of type A	98
3.2.1. Schur-Sergeev duality, I	98
3.2.2. Schur-Sergeev duality, II	100
3.2.3. The character formula	104
3.2.4. The classical Schur duality	105
3.2.5. Degree of atypicality of $\lambda^\natural$	106
3.2.6. Category of polynomial modules	108
§3.3. Representation theory of the algebra $\mathcal{H}_n$	109
3.3.1. A double cover	110
3.3.2. Split conjugacy classes in $\tilde{B}_n$	111
3.3.3. A ring structure on $R^-$	114
3.3.4. The characteristic map	116
3.3.5. The basic spin module	118
3.3.6. The irreducible characters	119
§3.4. Schur-Sergeev duality for $\mathfrak{q}(n)$	121
3.4.1. A double centralizer property	121
3.4.2. The Sergeev duality	123
3.4.3. The irreducible character formula	125
§3.5. Exercises	125
Notes	128
Chapter 4. Classical invariant theory	131
§4.1. FFT for the general linear Lie group	131
4.1.1. General invariant theory	132
4.1.2. Tensor and multilinear FFT for $GL(V)$	133
4.1.3. Formulation of the polynomial FFT for $GL(V)$	134
4.1.4. Polarization and restitution	135
§4.2. Polynomial FFT for classical groups	137
4.2.1. A reduction theorem of Weyl	137
4.2.2. The symplectic and orthogonal groups	139
4.2.3. Formulation of the polynomial FFT	140
4.2.4. From basic to general polynomial FFT	141
4.2.5. The basic case	142
§4.3. Tensor and supersymmetric FFT for classical groups	145
4.3.1. Tensor FFT for classical groups	145
4.3.2. From tensor FFT to supersymmetric FFT	147

§4.4. Exercises	149
Notes	150
Chapter 5. Howe duality	151
§5.1. Weyl-Clifford algebra and classical Lie superalgebras	152
5.1.1. Weyl-Clifford algebra	152
5.1.2. A filtration on Weyl-Clifford algebra	154
5.1.3. Relation to classical Lie superalgebras	155
5.1.4. A general duality theorem	157
5.1.5. A duality for Weyl-Clifford algebras	159
§5.2. Howe duality for type $A$ and type $Q$	160
5.2.1. Howe dual pair $(GL(k), gl(m n))$	160
5.2.2. $(GL(k), gl(m n))$ -Howe duality	162
5.2.3. Formulas for highest weight vectors	164
5.2.4. $(q(m), q(n))$ -Howe duality	166
§5.3. Howe duality for symplectic and orthogonal groups	169
5.3.1. Howe dual pair $(Sp(V), osp(2m 2n))$	170
5.3.2. $(Sp(V), osp(2m 2n))$ -Howe duality	172
5.3.3. Irreducible modules of $O(V)$	175
5.3.4. Howe dual pair $(O(k), spo(2m 2n))$	177
5.3.5. $(O(V), spo(2m 2n))$ -Howe duality	178
§5.4. Howe duality for infinite-dimensional Lie algebras	180
5.4.1. Lie algebras $\mathfrak{a}_\infty$ , $\mathfrak{c}_\infty$ , and $\mathfrak{d}_\infty$	180
5.4.2. The fermionic Fock space	183
5.4.3. $(GL(\ell), \mathfrak{a}_\infty)$ -Howe duality	184
5.4.4. $(Sp(k), \mathfrak{c}_\infty)$ -Howe duality	187
5.4.5. $(O(k), \mathfrak{d}_\infty)$ -Howe duality	190
§5.5. Character formula for Lie superalgebras	192
5.5.1. Characters for modules of Lie algebras $\mathfrak{c}_\infty$ and $\mathfrak{d}_\infty$	192
5.5.2. Characters of oscillator $osp(2m 2n)$ -modules	193
5.5.3. Characters for oscillator $spo(2m 2n)$ -modules	195
§5.6. Exercises	197
Notes	201
Chapter 6. Super duality	205
§6.1. Lie superalgebras of classical types	206
6.1.1. Head, tail, and master diagrams	206
6.1.2. The index sets	208
6.1.3. Infinite-rank Lie superalgebras	208
6.1.4. The case of $m = 0$	211
6.1.5. Finite-dimensional Lie superalgebras	213

6.1.6.	Central extensions	213
§6.2.	The module categories	214
6.2.1.	Category of polynomial modules revisited	215
6.2.2.	Parabolic subalgebras and dominant weights	217
6.2.3.	The categories $\mathcal{O}$ , $\overline{\mathcal{O}}$ , and $\widetilde{\mathcal{O}}$	218
6.2.4.	The categories $\mathcal{O}_n$ , $\overline{\mathcal{O}}_n$ , and $\widetilde{\mathcal{O}}_n$	220
6.2.5.	Truncation functors	221
§6.3.	The irreducible character formulas	222
6.3.1.	Two sequences of Borel subalgebras of $\widetilde{\mathfrak{g}}$	223
6.3.2.	Odd reflections and highest weight modules	225
6.3.3.	The functors $T$ and $\overline{T}$	228
6.3.4.	Character formulas	231
§6.4.	Kostant homology and KLV polynomials	232
6.4.1.	Homology and cohomology of Lie superalgebras	232
6.4.2.	Kostant $u^-$ -homology and $u$ -cohomology	235
6.4.3.	Comparison of Kostant homology groups	236
6.4.4.	Kazhdan-Lusztig-Vogan (KLV) polynomials	239
6.4.5.	Stability of KLV polynomials	240
§6.5.	Super duality as an equivalence of categories	241
6.5.1.	Extensions à la Baer-Yoneda	241
6.5.2.	Relating extensions in $\mathcal{O}$ , $\overline{\mathcal{O}}$ , and $\widetilde{\mathcal{O}}$	243
6.5.3.	Categories $\mathcal{O}^f$ , $\overline{\mathcal{O}}^f$ , and $\widetilde{\mathcal{O}}^f$	247
6.5.4.	Lifting highest weight modules	247
6.5.5.	Super duality and strategy of proof	248
6.5.6.	The proof of super duality	250
§6.6.	Exercises	255
	Notes	258
Appendix A.	Symmetric functions	261
§A.1.	The ring $\Lambda$ and Schur functions	261
A.1.1.	The ring $\Lambda$	261
A.1.2.	Schur functions	265
A.1.3.	Skew Schur functions	268
A.1.4.	The Frobenius characteristic map	270
§A.2.	Supersymmetric polynomials	271
A.2.1.	The ring of supersymmetric polynomials	271
A.2.2.	Super Schur functions	273
§A.3.	The ring $\Gamma$ and Schur $Q$ -functions	275
A.3.1.	The ring $\Gamma$	275
A.3.2.	Schur $Q$ -functions	277
A.3.3.	Inner product on $\Gamma$	278

---

A.3.4. A characterization of $\Gamma$	280
A.3.5. Relating $\Lambda$ and $\Gamma$	281
§A.4. The Boson-Fermion correspondence	282
A.4.1. The Maya diagrams	282
A.4.2. Partitions	282
A.4.3. Fermions and fermionic Fock space	284
A.4.4. Charge and energy	286
A.4.5. From Bosons to Fermions	287
A.4.6. Fermions and Schur functions	289
A.4.7. Jacobi triple product identity	289
Notes	290
Bibliography	291
Index	299

# Lie superalgebra ABC

We start by introducing the basic notions and definitions in the theory of Lie superalgebras, such as basic and queer Lie superalgebras, Cartan and Borel subalgebras, root systems, positive and fundamental systems. We formulate the main structure results for the basic Lie superalgebras and the queer Lie superalgebras. We describe in detail the structures of Lie superalgebras of type  $gl$ ,  $osp$  and  $q$ . A distinguishing feature for Lie superalgebras is that Borel subalgebras, positive systems, or fundamental systems of a simple finite-dimensional Lie superalgebra may not be conjugate under the action of the corresponding Weyl group; rather, they are shown to be related to each other by real and odd reflections. A highest weight theory is developed for Lie superalgebras. We describe how fundamental systems are related and how highest weights are transformed by an odd reflection.

## 1.1. Lie superalgebras: Definitions and examples

Throughout this book we will work over the field  $\mathbb{C}$  of complex numbers. Let

$$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

denote the group of two elements and let  $\mathfrak{S}_n$  denote the symmetric group in  $n$  letters.

In this section, we introduce many examples of Lie superalgebras. The examples, most relevant to this book, of the general linear and ortho-symplectic Lie superalgebras are introduced first. Other series of finite-dimensional simple Lie