MATRICES AND LINEAR PROGRAMMING
with applications

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Preface

Over the past few decades, both the use and the importance of quantitative approaches have increased, not only in science and engineering, but also such fields as business, management, economics, information science, and the social sciences. In addition, the widespread use of computers in these disciplines has further accelerated the importance of quantitative technique.

Because quantitative approaches have become so significant, most colleges and universities now require students majoring in the above-mentioned fields to complete courses in three mathematical areas: matrix algebra (possibly with introductory linear programming); calculus; and probability–statistics. This book has been carefully designed to meet the needs of the first area—especially one-semester courses in matrix algebra and introductory linear programming.

Courses for which this text is aimed are usually taught by mathematics departments as one-semester undergraduate courses. These go by many names, including matrices and linear programming, matrices and applications, matrix algebra, linear algebra, and finite mathematics. Or, in some institutions, the materials may be covered in one of a series of basic mathematics courses called Mathematical Concepts I, II, III, and so on. Although the book is written primarily for undergraduates at the sophomore or junior level, it is also suitable for seniors and first-year graduate students.

Whereas the importance of matrices and linear programming is not in doubt, we may raise questions concerning the differing methods of *how* to teach these subjects. We could, for instance, elect to teach abstract theoretical mathematics, with few discussions on practical applications. This approach, however, might seriously confuse many nonmath majors. We could also take what might be termed a "cookbook" approach wherein no emphasis at all is placed on *understanding* mathematics. These instructional procedures represent the extremes, and the method employed in this book occupies a middle ground between them. Thus, this text is tailored to fit the needs of those large number of nonscience majors who require a background in matrix algebra.

The content has, therefore, been structured to include the following advantageous features:

1. High school algebra is the only prerequisite necessary. Yet the book should provide a sound background for practical applications as well as for more advanced studies.
2. Easy-to-understand, practical applications receive emphasis. No stress is placed on rigorous and abstract mathematical theories. Many problems and comments involving practical applications are presented throughout. The Certified Public Accountant (CPA) exam questions, for instance, furnish excellent application examples of the materials discussed. The variety of problems and comments are valuable not only for an understanding of how the materials could be used to solve real-world problems, but—more important—these illustrations make the mathematics even more thoroughly comprehensible.
3. Comments for computer-related applications are included. Today, access to computers is commonplace and this access will be considerably greater ten or twenty years from now. For this reason, computer-related materials have been inserted. These do not, however, interrupt the mainstream of the book.

Probability concepts are not included except in sections dealing with game theory and Markov chains. Sections, topics, and exercises preceded by the dagger symbol (†) are optional. In addition to these optional materials, the following sections may be skipped when less emphasis is placed upon theoretical aspects: Sections 2.5, 3.6, 4.6, 5.3, and 7.5. When only matrix algebra is being taught, Part 2 can be skipped without loss of continuity.

A course using this book can be taught either before, after, or concurrently with calculus and/or probability–statistics courses.

The materials in this text were successfully used and refined at Virginia State College, Petersburg, Virginia, for six years in quantitative methods courses consisting of students majoring in business administration, accounting, and business information systems.

The author would like to acknowledge several people, especially E. P. Winkofsky, Virginia Polytechnic Institute and State University, and James F. Hurley, University of Connecticut, for their invaluable comments. The author would also like to express his appreciation to his colleagues and students at Virginia State College for correcting grammatical errors in the manuscript. He is especially indebted to Dr. Donald L. Campbell for improving the readibility of the book. He also wishes to thank Mrs. Jacqueline E. Goode for typing the manuscript.

Toshinori Munakata

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part 1

MATRICES

1

Fundamentals of Matrices

1.1 WHAT IS A MATRIX?

1. What It Looks Like

Here is an example of a *matrix*:

$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

A **matrix** is a *rectangular array of mathematical **elements*** such as numbers, variables, and functions; it is usually enclosed by a pair of brackets.

In this first example, the number 2 is an element, 0 is an element, and so on. This matrix contains six elements.

The following is another example of a matrix.

$$\begin{bmatrix} 0.5 \\ x \\ x^2 \\ \log x \end{bmatrix}$$

In this matrix, 0.5 is a number, x is a variable, and x^2 and $\log x$ are functions of x.

In the first example there are two rows and three columns.

$$\begin{array}{cc} & \text{Column} \quad \text{Column} \quad \text{Column} \\ & \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \text{Row} \longrightarrow & \left[\begin{array}{ccc} 2 & 0 & 1 \\ -1 & 3 & 2 \end{array}\right] \\ \text{Row} \longrightarrow & \end{array}$$

This first example is called a (2×3) matrix (read "two by three matrix") or is said to be of order 2 by 3. That is, the **order** of a matrix is (the number of rows) by (the number of columns). The order of the second example is 4 by 1. (Note that it is not 1 by 4.)

A matrix with m rows and n columns—that is, an $(m \times n)$ matrix—can generally be written as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In this form, $a_{11}, a_{12}, \cdots$ are the general representation of the elements. For example, a_{11} can be a number, or a variable, or a function. Notice that the first subscript shows the row of the element, and the second subscript shows the column. For example, a_{23} is the element of the second row and the third column. Similarly, a_{mn} is the element of the m-th row and n-th column.

This general representation of matrix elements can be used to refer to a specific element in the matrix. For instance, the (-1) element in our first example can be referred to as a_{21}.

2. Short Forms of Matrices

A matrix, according to our definition, is a rectangular array of elements such as numbers, variables, or functions. It requires a large space and a number of written elements. How large a space it needs depends on the order of the matrix. Sometimes a short form of a matrix is desirable.

The short form is useful for several reasons. First of all, the same matrix may sometimes be referred to time and time again, and the short form makes this repeated referencing much easier.

Secondly, the short form is better because sometimes the specific values of the elements or the order of the matrix is immaterial. This condition may occur in general discussions of matrix algebra or in a computer program. The order of the matrix and specific values of the elements are fed into a computer at a later stage, but do not have to be specified at early stages.

Under such circumstances, a matrix may be written in the short form

$$[a_{ij}]_{2 \times 3}$$

The expression 2×3 indicates that this is an order (2×3) matrix. An $(m \times n)$ matrix may also be written in the short form

$$[a_{ij}]_{m \times n}$$

The expression a_{ij} is the i-th row, j-th column element and it is a general representation of an element of the matrix. Further, when the order of the matrix has been previously specified and is understood, or when it is immaterial, the matrix may be written without the order specification as

$$[a_{ij}]$$

In general, this is not a (1×1) matrix. If it were, it would be

$$[a_{11}]$$

Another short and convenient form may be used to represent a matrix. This form appears as

$$A$$

This single character form is a "symbolic" representation of a matrix. Usually, italic capital letters are used for symbolic representations of matrices, like matrices A, B, and C, or X and Y. Note that although A is a single character, it does not represent a single value (say $A = 5$) like a variable in high school algebra. Instead, it represents a whole matrix as a "package," which usually includes many elements. For example, a matrix A may be

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

and a matrix B may be

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

Note that in the second example, the expression b_{ij} is used instead of the a_{ij} expression. The reason for this is that they are elements of matrix B. Similarly, the elements of a matrix X will be $x_{11}, x_{12}, \cdots$.

3. Matrices—A Great Mathematical Invention

The history of mathematics is filled with many great inventions. Matrices certainly stand as one. The heart of the matrix invention is its single-character symbolic form, which we have just discussed. This symbolic form is not only a convenient, abbreviated way to represent a matrix, but is, indeed, the very core of this invention.

> When a single character is used to represent a matrix, that matrix can then by symbolically treated as a *single object.*

You can add two or more matrices just as you can add two or more numbers. You can multiply matrices, define different kinds of matrices, and perform a number of other operations. Various properties and theorems on matrices and their operations can be derived. The point is that, in most cases, these properties and theorems can be represented in symbolic form (such as $AB + AC = A(B + C)$). Without going to every element of the matrix in every step, you can manipulate symbolic matrices just as you can manipulate high school algebra variables to solve problems. This process makes the evaluation of solutions simple and clear. This is the essence of **matrix algebra**.

Matrix algebra has increased its importance in conjunction with the use of computers.

> Matrix algebra can be easily adapted for computer use, and conversely, many computer applications use the concept of matrices.

In some computer languages (such as BASIC or APL), symbolic matrix notations and manipulations are allowed, in which case matrix algebra fits extremely well.

4. How Matrices Are Related to the Real World

Table 1-1 shows the volume of students enrolled in different schools in a university.

Table 1-1

Number of Enrollees in Different Schools in a University

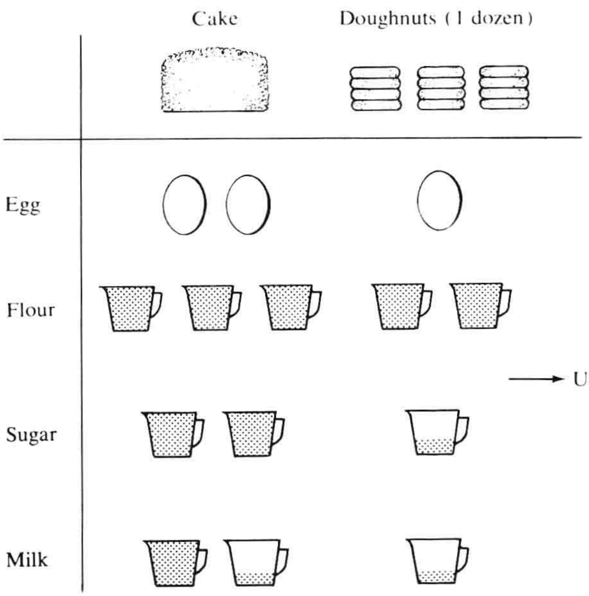
	Arts and Sciences	Business	Education	Engineering	Medicine
Freshman	962	1054	858	543	0
Sophomore	878	985	895	452	0
Junior	685	880	854	350	153
Senior	650	789	730	329	142

This table, which is a rectangular array of numbers, might be called a matrix, although in conventional matrix form it is usually rewritten as

	Arts and Sciences	Business	Education	Engineering	Medicine
Freshman	962	1054	858	543	0
Sophomore	878	985	895	452	0
Junior	685	880	854	350	153
Senior	650	789	730	329	142

We do not have to be too rigorous about the appearance of a matrix. In the above example, either form may be considered as a matrix, although the second form is conventional.

Let us take a look at another example.



→ U =

	Cake	Doughnuts	
	2	1	Egg
	3	2	Flour (cup)
	2	$\frac{1}{3}$	Sugar (cup)
	$1\frac{1}{4}$	$\frac{1}{4}$	Milk (cup)

This matrix—*U*—shows that 2 eggs, 3 cups of flour, 2 cups of sugar and 1-1/4 cups of milk are used to make one cake.

As you can see from these examples, innumerable kinds of data can be represented in matrix form. The key idea is that each row and column correspond to certain classifications. The order of a matrix can be small or large. If this student enrollment matrix is extended to specify the number of students in different majors instead of schools, then the order of the matrix may be (4×50), assuming there are 50 different majors in the university.

After these matrices are constructed from raw data, they may be manipulated in various ways to determine various quantities. For example, the enrollment matrix may be multiplied by another matrix, one that might give total budget and graduate assistantship needs.

Matrix algebra is important for understanding and effectively applying various techniques of matrix operations used to solve problems. In Part 1 we will discuss the fundamentals of matrix algebra.

5. More on How Matrices Are Related to the Real World

As we have mentioned, matrices are derived from many kinds of data in the real world. Sometimes certain kinds of data lead to a system (or set) of simultaneous linear equations, often simply called a *set of linear equations.* A matrix may also be derived from a set of linear equations.

Consider the set of equations

$$\left.\begin{aligned} -5x + 4y + 6z &= 12 \\ 3x - 2y + z &= 7 \\ x \qquad - z &= -1 \end{aligned}\right\}$$

Since a variable x always appears in the first column, (y always appears in the second column, and so on), it is not always essential to write these variables. This suggests that we may write the left-hand side of the above equations in compact matrix form:

$$\begin{bmatrix} -5 & 4 & 6 \\ 3 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Note that x is equal to $1x$; therefore, 1 is the corresponding entry of x. Note also that the 0 entry indicates that there is no y term in the third equation (no y term is equivalent to $0y$).

This simple matrix representation of the set of equations is quite all right as long as you understand its meaning—the first column elements represent the coefficients of x, the first row elements represent the coefficients of the first equation, and so on. Such a matrix is referred to as a **coefficient matrix**.

Sometimes the constants on the right-hand side of a set of equations are also included. In the example under discussion, the matrix will be:

$$\begin{bmatrix} -5 & 4 & 6 & 12 \\ 3 & -2 & 1 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

Sometimes a vertical dotted line is placed to separate the entries on the right- and left-hand sides, as

$$\left[\begin{array}{ccc:c} -5 & 4 & 6 & 12 \\ 3 & -2 & 1 & 7 \\ 1 & 0 & -1 & -1 \end{array}\right]$$