A COURSE IN

Linear Algebra

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Harcourt Brace Jovanovich, Publishers
and its subsidiary, Academic Press

San Diego New York Chicago Austin Washington, D.C.
London Sydney Tokyo Toronto

ISBN: 0-15-515134-7
Library of Congress Catalog Card Number: 87-81875

Printed in the United States of America

Preface

A Course in Linear Algebra was shaped by three major aims: (1) to present a reasonably complete, mathematically honest introduction to the basic concepts of linear algebra; (2) to raise our students' level of mathematical maturity significantly in the crucial sophomore year of their undergraduate careers; and, (3) to present the material in an interesting, inspiring fashion, to provide motivation for the students and give them a feeling both for the overall logical structure of the subject and for the way linear algebra helps to explain phenomena and solve problems in many areas of mathematics and its applications.

We firmly believe that, even today, the traditional theoretical approach to the teaching of linear algebra is appropriate and desirable for students who intend to continue their studies in the mathematical sciences.

With such goals in mind, we have chosen to present linear algebra as the study of vector spaces and linear mappings between vector spaces rather than as the study of matrix algebra or the study of numerical techniques for solving systems of linear equations. We define general vector spaces and linear mappings at the outset, and we base all of the subsequent developments on these ideas.

We feel that this approach has several major benefits for our intended audience of mathematics majors (and for others as well). First, it highlights the way seemingly unrelated sets of phenomena, such as the algebraic properties of vectors in the plane and the algebraic properties of functions $f:\mathbf{R} \to \mathbf{R}$, or the geometric behavior of projections and rotations in the plane and the differentiation rules for sums and scalar multiples of functions, may be unified and understood as different instances of more fundamental patterns. Furthermore, once these essential similarities are recognized, they may be exploited to solve other problems.

Second, our approach provides a ready-made *context, motivation,* and *geometric interpretation* for each new computational technique that we present. For example, the Gauss-Jordan elimination procedure for solving systems of linear equations is introduced first in order to allow us to answer questions about parametrizations (spanning sets) for subspaces, linear independence and dependence of sets of vectors, and the like.

Finally, our approach offers the opportunity to introduce proofs and abstract problem-solving into the course from the beginning. We believe that all students of mathematics at this level must begin to practice applying what they have learned in new situations, and not merely master routine calculations. In addition, they must begin to learn how to construct correct and convincing proofs of their assertions—that is the way they will be working and communicating with their colleagues as long as they stay in the mathematical sciences. Since the subject matter of linear algebra is relatively uncomplicated, this is the ideal place to start.

We have included some important mathematical applications of the topics that we cover, such as the application of the spectral theorem for symmetric real matrices to the geometry of conic sections and quadric surfaces, and the application of diagonalization and Jordan canonical form to the theory of systems of ordinary differential equations. We have not included many applications of linear algebra to problems in other disciplines, however, both because of the difficulty of presenting convincing, realistic applied problems at this level, and because of the needs of our audience. We prefer to give students a deep understanding of the mathematics that will be useful and to leave the discussion of the applications themselves to other courses.

A WORD ON PREREQUISITES

Since students taking the sophomore linear algebra course have typically had at least one year of one-variable calculus, we have felt free to use various facts from calculus (mainly properties of and formulas for derivatives and integrals) in many of the examples and exercises in the first six chapters. These examples may be omitted if necessary. The seventh chapter, which covers some topics in differential equations, uses substantially more calculus, through derivatives of vector-valued functions of one variable and partial derivatives.

In the text and in proofs, we have also freely used some ideas such as the technique of proof by mathematical induction, the division algorithm for polynomials in one variable and its consequences about roots and factorizations of polynomials, and various notions about sets and functions. Ideally, these topics should be familiar to students from high school mathematics; they are also reviewed briefly in the text or in the appendices for easy reference.

SOME COMMENTS ON THE EXERCISES

In keeping with our goals for the course, we have tried to structure the book so that, as they progress through the course, students will start to become active participants in the theoretical development of linear algebra, rather than remaining passive bystanders. Thus, in the exercises for each section, in addition to computational problems illustrating the main points of the section, we have included proofs of parts of propositions stated in the text and other problems dealing with related topics and extensions of the results of the text. We have sometimes used the exercises to introduce new ideas that will be used later, when those ideas are straightforward enough to make it possible to ask reasonable questions with only a minimum of background. In addition to the exercises at the end of each section, there are supplementary exercises at the end of each chapter. These provide a review of the foregoing material and extend concepts developed in the chapter. Included in the supplementary exercises are true-false questions designed to test the student's mastery of definitions and statements of theorems.

Following the appendices, we provide solutions to selected exercises. In particular, we give solutions to alternative parts of exercises requiring numerical solutions, solutions to exercises that are proofs of propositions in the text, and solutions to exercises that are used subsequently in the text.

Finally, as the course progresses, we have included numerous extended sequences of exercises that develop other important topics in, or applications of, linear algebra. We strongly recommend that instructors using this book assign some of these exercises from time to time. Though some of them are rather difficult, we have tried to structure the questions to lead students to the right approach to the solution. In addition, many hints are provided. These exercise sequences can also serve as the basis for individual or group papers or in-class presentations, if the instructor desires. We have found assignments of this type to be very worthwhile and enjoyable, even for students at the sophomore level. A list of these exercises appears after the acknowledgments.

ACKNOWLEDGMENTS

We would first like to thank our students at Holy Cross for their participation and interest in the development of this book. Their comments and suggestions have been most valuable at all stages of the project. We thank our colleagues Mel Tews, Pat Shanahan, and Margaret Freije, who have also taught from our material and have made numerous recommendations for improvements that have been incorporated in the book. The following reviewers have read the manuscript and contributed many useful comments: Susan C. Geller, Texas A&M University; Harvey J. Schmidt, Jr., Lewis and Clark College; Joseph Neggers, University of Alabama; Harvey C. Greenwald, California Polytechnic State University; and Samuel G. Councilman, California State University, Long Beach.

Finally, we thank Mrs. Joy Bousquet for her excellent typing, and the entire production staff at Harcourt Brace Jovanovich.

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COURSES OF STUDY

To allow for flexibility in constructing courses using this book, we have included more material than can be covered in a typical one-semester course. In fact, the book as a whole contains almost exactly enough material for a full-year course in linear algebra. The basic material that should be covered in any one-semester course is contained in the first three chapters and the first two sections of the fourth chapter. We have structured the book so that several different types of coherent one-semester courses may be constructed by choosing some additional topics and omitting others.

Option I

A one-semester course culminating in the geometry of the inner product space $\mathbf{R}^n$ and the spectral theorem for symmetric mappings would consist of the core material plus the remaining sections of Chapter 4. We have covered all of this material in a 13-week semester at Holy Cross.

Option II

A one-semester course culminating in an introduction to differential equations would consist of the core material plus Sections 1, 2, 4, and 5 of Chapter 7. (The third section of Chapter 7 presupposes the Jordan canonical form, and the description of the form of the solutions to higher-order, constant-coefficient equations in Section 4 is deduced by reducing to a system of first-order equations and applying the results of Section 3. Nevertheless, the main result of Section 4, Theorem 7.4.5, could be justified in other ways, if desired. The Jordan form could also be introduced without proof in order to deduce these results, since the final method presented for explicitly solving the equations makes no use of matrices.)

Option III

A one-semester course incorporating complex arithmetic and the study of vector spaces over **C** and other fields would consist of the core material plus Sections 1 and 2 of Chapter 5. (If time permits, additional sections from Chapter 4 or Chapter 6 could also be included.)

A Guide to the Exercises

Below is a listing by topic of the exercises that either introduce a new topic used later in the text or develop important extensions of the material covered in the text. The topics are listed in order of their first occurrence.

Topic	*Section (Exercise)*
Even and odd functions	1.2 (2)
Vector spaces of matrices	1.2 (11–16)
	1.3 (11–12)
	2.2 (10–15)
Direct sums of subspaces	1.3 (8–9)
Alternate version of elimination	1.5 (7–8)
Lagrange interpolation	1.6 (16)
Vector space of linear maps	2.1 (12)
	2.2 (10–15)
Restriction of a linear transformation	2.1 (14–15)
	2.6 (9)
Reflections	2.2 (4)
	2.7 (14)
	4.4 (14)
Transposes, symmetric, and skew-symmetric matrices	2.2 (12–13)
	2.3 (9)
Elimination in matrix form	2.3 (10–11)
Traces of matrix	2.3 (12)
Transpose as a linear transformation	2.3 (13)
Direct sum of linear transformations	2.4 (11)
	6.1 (6)

Topic	*Section (Exercise)*
Invariant subspace of set of mappings	6.4 (7)
Irreducible set of mappings	6.4 (8–9)
Markov chains	6.4 (12)
Finite difference equations	7.2 (8–12)
	7.3 (15–18)
	7.4 (14–15)
Inhomogeneous first order ordinary differential equations	7.4 (8)
Alternate method of solving linear differential equations	7.4 (9,10)
Ordinary differential equations as operators	7.4 (11)
Non-constant coefficient ordinary differential equations	7.4 (12)
Heat equation	7.5 (12)
Legendre polynomials	7, Supplementary Exercises (7–12)

Contents

CHAPTER 1

Vector Spaces

Introduction

In this first chapter of *A Course in Linear Algebra,* we begin by introducing the fundamental concept of a vector space, a mathematical structure that has proved to be very useful in describing some of the common features of important mathematical objects such as the set of vectors in the plane and the set of all functions from the real line to itself.

Our first goal will be to write down a list of properties that hold for the algebraic sum and scalar multiplication operations in the two aforementioned examples. We will then take this list of properties as our definition of what a general vector space should be. This is a typical example of the idea of defining an object by specifying what properties it should have, a commonly used notion in mathematics.

We will then develop a repertoire of examples of vector spaces, drawing on ideas from geometry and calculus. Following this, we will explore the inner structure of vector spaces by studying subspaces and spanning sets and bases (special subsets from which the whole vector space can be built up). Along the way, we will find that most of the calculations that we need to perform involve solving simultaneous systems of linear equations, so we will also discuss a general method for doing this.

The concept of a vector space provides a way to organize, explain, and build on many topics you have seen before in geometry, algebra, and calculus. At the same time, as you begin to study linear algebra, you may find that the way everything is presented seems very general and abstract. Of course, to the extent that this is true, it is a reflection of the fact that mathematicians have seen a very basic general pattern that holds in many different situations. They have exploited this information by inventing the ideas discussed in this chapter in order to understand all these situations and treat them all without resorting to dealing with each case separately. With time and practice, working in general vector spaces should become natural to you, just as the equally abstract concept of number (as opposed to specific collections of some number of objects) has become second nature.

§1.1. VECTOR SPACES

The basic geometric objects that are studied in linear algebra are called vector spaces. Since you have probably seen vectors before in your mathematical experience, we begin by recalling some basic facts about vectors in the plane to help motivate the discussion of general vector spaces that follows.

In the geometry of the Euclidean plane, a *vector* is usually defined as a directed line segment or "arrow," that is, as a line segment with one endpoint distinguished as the "head" or final point, and the other distinguished as the "tail" or initial point. See Figure 1.1. Vectors are useful for describing quantities with both a magnitude and a direction. Geometrically, the *length* of the directed line segment may be taken to represent the magnitude of the quantity; the direction is given by the direction that the arrow is pointing. Important examples of quantities of this kind are the instantaneous velocity and the instantaneous acceleration at each time of an object moving along a path in the plane, the momentum of the moving object, forces, and so on. In physics these quantities are treated mathematically by using vectors as just described.

In linear algebra one of our major concerns will be the *algebraic properties* of vectors. By this we mean, for example, the operations by which vectors may be combined to produce new vectors and the properties of those operations. For instance, if we consider the set of *all* vectors in the plane with a tail at some fixed point O, then it is possible to combine vectors to produce new vectors in two ways.

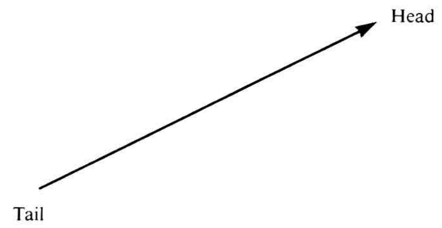

Figure 1.1

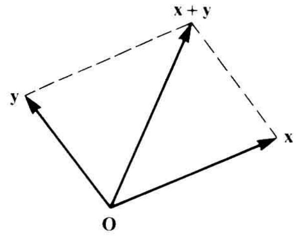

Figure 1.2

First, if we take two vectors **x** and **y**, then we can define their *vector sum* **x** + **y** to be the vector whose tail is at the point O and whose head is at the fourth corner of the parallelogram with sides **x** and **y**. See Figure 1.2. One physical interpretation of this sum operation is as follows. If two forces, represented by vectors **x** and **y**, act on an object located at the point O then the resulting force will be given by the vector sum **x** + **y**.

Second, if we take a vector **x** and a positive real number c (called a *scalar* in this context), then we can define the *product* of the vector **x** and the scalar c to be the vector in the same direction as **x** but with a magnitude or length that is equal to c times the magnitude of **x**. If $c > 1$, this has the effect of magnifying **x**, whereas if $c < 1$, this shrinks **x**. The case $c > 1$ is pictured in Figure 1.3. Physically, a positive scalar multiple of a vector may be thought of in the following way. For example, in the case $c = 2$, if the vector **x** represents a force, then the vector 2**x** represents a force that is "twice as strong" and that acts in the same direction. Similarly, the vector (1/2)**x** represents a force that is "one-half as strong." The product c**x** may also be defined if $c < 0$. In this case the vector c**x** will point along the same line through the origin as **x** but in the opposite direction from **x**. The magnitude of c**x** in this case will be equal to $|c|$ times the magnitude of **x**. See Figure 1.4.

Further properties of these two operations on vectors may be derived directly from these geometric definitions. However, to bring their algebraic nature into clearer focus, we will now consider an alternate way to understand these operations. If we introduce the familiar Cartesian coordinate system in the plane and place the origin at the point O = (0, 0), then a vector whose tail is at O is uniquely specified

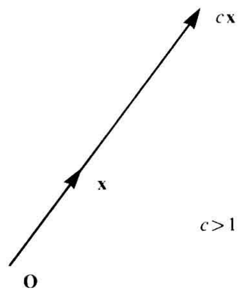

Figure 1.3

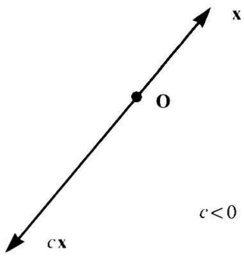

Figure 1.4

by the coordinates of its head. That is, vectors may be described as ordered pairs of real numbers. See Figure 1.5.

In this way we obtain a one-to-one correspondence between the set of vectors with a tail at the origin and the set $\mathbf{R}^2$ (the set of ordered pairs of real numbers) and we write $\mathbf{x} = (x_1, x_2)$ to indicate the vector whose head is at the point (x_1, x_2).

Our two operations on vectors may be described using coordinates. First, from the parallelogram law for the vector sum, we see that if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$. See Figure 1.6. That is, to find the vector sum, we simply add "component-wise." For example the vector sum $(2, 5) + (4, -3)$ is equal to $(6, 2)$. Second, if c is a scalar and $\mathbf{x} = (x_1, x_2)$ is a vector, then $c\mathbf{x} = (cx_1, cx_2)$. The scalar multiple $4(-1, 2)$ is equal to $(-4, 8)$.

With this description, the familiar properties of addition and multiplication of real numbers may be used to show that our two operations on vectors have the following algebraic properties:

1. The vector sum is *associative:* For all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} \in \mathbf{R}^2$ we have

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

2. The vector sum is *commutative:* For all $\mathbf{x}$ and $\mathbf{y} \in \mathbf{R}^2$ we have

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

3. There is an *additive identity element* $\mathbf{0} = (0, 0) \in \mathbf{R}^2$ with the property that for all $\mathbf{x} \in \mathbf{R}^2$, $\mathbf{x} + \mathbf{0} = \mathbf{x}$.

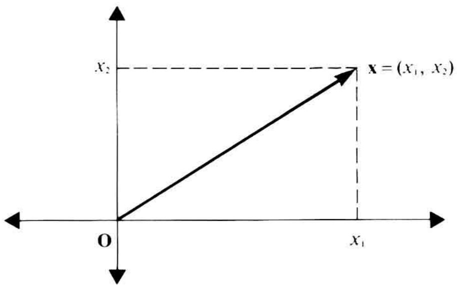

Figure 1.5