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Finite Geometries and Designs

Proceedings of the Second Isle of Thorns Conference 1980

Edited by P. J. CAMERON, J. W. P. HIRSCHFELD & D. R. HUGHES

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PREFACE

This book contains the proceedings of a conference that took place from 15th to 19th June, 1980, at the White House, Isle of Thorns, Chelwood Gate, England, which is a conference centre run by the University of Sussex.

There are 36 articles in all. The Introduction provides a background for non-specialists and places in context the 35 papers submitted. An asterisk following an author's name in some multi-authored papers indicates the presenter of the paper. At the back of the book are a list of talks given for which there is no paper and a list of participants.

We are indebted to the British Council for supporting some of the participants and to the publishers John Wiley, McGraw Hill, Oxford University Press, Pitman and Springer for supporting a book exhibition.

Above all, we are profoundly grateful to Mrs. Jill Foster of the University of Sussex for the excellent typing of the camera-ready copy from which this book was produced.

P.J.C.
J.W.P.H.
D.R.H.

CONTENTS

INTRODUCTION

One of the features of geometry, and of finite geometry in particular, is the difficulty of giving a concise definition of the subject. As well as the wide variety of structures that are studied and techniques that are used, an important factor contributing to this intractibility is the way in which different parts of the subject link up with and influence one another. This is part of the excitement of the subject for its practitioners, but may be off-putting for outsiders who see a confused tangle rather than an elegant network. The purpose of this introduction is to attempt to trace some of the main threads of finite geometry, and to locate the papers of this collection in the warp and weft of its fabric.

The structure of the subject militates against a linear tour of its highlights; but, of course, there is no other way to write an introduction! To simplify the task, we regard finite projective geometries (Galois spaces) as the central concept.

Let n be a positive integer, and q a prime power; let $GF(q)$ denote the Galois field with q elements. The elements of the n-dimensional *projective geometry* $PG(n,q)$ or $S_{n,q}$ are the subspaces of an $(n+1)$-dimensional vector space V over $GF(q)$; each has a geometric dimension which is one less than its vector space dimension. Thus the basic objects, the points, are the 1-dimensional subspaces of V. It is common to identify an arbitrary subspace with the set of points it contains.

We may loosely divide the study of finite projective geometry into two parts, whose extensions and relations cover a great part of combinatorics : characterisations, and the study of subsets. We deal with these in turn.

Axioms for projective spaces are "classical", and can be found in Veblen and Young's "Projective Geometry". In terms of points and lines, they may be stated as follows:

(i) any line has at least three points;

(ii) any two points lie on a unique line;

(iii) a transversal to two sides of a triangle meets the third side also.

A subspace can be defined as a set of points containing the line through any two of its points; and the dimension of the geometry is the number of subspaces in a maximal chain (excluding the empty set and the whole space).

The characterisation has a very important feature. Any finite structure satisfying the axioms and having dimension at least 3 is isomorphic to PG(n,q) for some n and q ; but this is not so for dimension 2 . Here, *projective planes* not isomorphic to the "classical" PG(2,q) exist. They are charcterised by the first two of the above axioms together with the requirement that any two lines are concurrent. The planes PG(2,q) are called *Desarguesian,* since they are characterised by the additional requirement that the theorem of Desargues is valid.

A finite projective plane has an *order* n , with the property that any line has $n+1$ points, and there are n^2+n+1 points altogether. The order of PG(2,q) is q .

If a line of a projective plane and all its points are deleted, the resulting structure is called an *affine plane.* Its lines fall into parallel classes in such a way that Euclid's parallel postulate holds. The projective plane can be reconstructed from the affine plane by adjoining a "point at infinity" to the lines of each parallel class, the new points lying on a "line at infinity". In a similar way, the *affine space* AG(n,q) is obtained from PG(n,q) by deleting a hyperplane (a subspace of codimension 1) together with all the non-empty subspaces it contains.

The study of finite projective and affine planes is of enormous importance in finite geometry; much of Dembowski's compendious "Finite Geometries" is devoted to this topic. The principal tool has been the use of collineation groups, and the interplay between groups and planes has provided a lot of information. Four papers in the present collection, those by Cohen *et al.*, Foulser, Hering and Kallaher, develop this theme.

Prominent in such studies is the notion of a *central collineation*, one which fixes all the lines through a point (called its *centre*). Such a collineation also has the property that it fixes all the points on a line (called the *axis*); it is called an *elation* or a *homology* according as the centre lies on the axis or not. If we choose the axis as line at infinity, then elations and homologies are translations and dilatations respectively of the affine plane, fixing every parallel class.

A typical example of the role of central collineations in characterisation theorems is the following well-known result. It is easy to see that, in a plane of order n, there are at most n elations with given centre and axis. If every incident point and line are centre and axis for n elations, then the plane is necessarily Desarguesian.

Hering shows that, if an affine plane possesses central collineations of certain types, then its full collineation group has a unique minimal normal subgroup, which is a simple group. Thus it is to be expected that the effect of recent work on the classification of finite simple groups will be felt in this part of geometry: we should examine the known simple groups to see how they can act on planes. Both Hering and Kallaher take up this theme.

In a projective plane, an involution (a collineation of order 2) which is not central is called a *Baer involution*, and has the property that its fixed points and lines form a subplane whose order is the square root of the order of the plane (a *Baer subplane*). (In general, the order of a subplane of a projective plane of order n cannot exceed $\sqrt{n}$. However, there can be affine subplanes of larger order: for example, both PG(2,4) and PG(2,7) "contain" AG(2,3). This situation is studied by Vedder.)

A feature of affine spaces is the existence of a group of translations, acting trasitively on the points but fixing every parallel class. The most important affine planes, the translation planes (studied by Cohen *et al.*, Foulser and Kallaher) have the same property. Cohen, Ganley and Jha consider translation planes admitting a collineation group fixing a subplane and acting transitively on the parallel classes outside this subplane; Kallaher deletes the transitivity condition but assumes that the group is of known type; while Foulser takes the specific case where the plane has order 81 but assumes only the existence of two collineations of order 3 whose fixed point sets are overlapping Baer subplanes.

Other classes of structures admitting translations are considered by Jungnickel and Marchi. Among Jungnickel's structures are the class-regular Hjelmslev planes, studied further by Sane.

A large area of combinatorics, the theory of designs, involves a generalisation of the first two axioms for projective geometries. If t is a positive integer, a t-*design* consists of a set of points equipped with a collection of proper subsets called "blocks", the blocks having a constant size k, and any t points lying in a non-zero constant number λ of blocks. If $\lambda = 1$, the design is called a *Steiner system*. Thus, the subspaces of fixed dimension in a projective space are the blocks of a 2-design; the lines form a Steiner system.

As yet, no non-trivial t-design with $t \geq 6$ is known; but there are two remarkable 5-designs, both Steiner systems, having 12 and 24 points, which have been known since the 1930s. These designs are connected with Mathieu's simple groups and Golay's perfect error-correcting codes, and have been intensively studied; yet this remarkable seam has further wealth to yield. Beth gives a new construction and uniqueness proof for the first of them.

The design formed by the points and hyperplanes of $PG(n,q)$ has the property that the number of points and blocks are equal. More generally, any 2-design with this property is called *symmetric*. The symmetric designs are extremal with respect to *Fisher's inequality*, which asserts that a 2-design has at least as many blocks as points.

Biggs and Ito describe another situation in which extremal configurations are often provided by symmetric designs. An ordinary (undirected) regular graph with girth 6 and valency k has at least $2(k^2 - k + 1)$ vertices. Equality is attained if and only if the graph can be constructed as follows: the vertices are the points and lines of a projective plane of order $k - 1$, two vertices being adjacent exactly when they correspond to a point and a line which are incident in the plane. A similar construction, applied to a symmetric design with $\lambda > 1$, would yield a graph of girth 4; yet this graph may have a λ-fold covering graph whose girth is 6. If such a covering graph exists, the number of its vertices exceeds the bound by just $2(\lambda - 1)$. Several examples exist.

A recent approach to design theory has made use of error-correcting codes. The setting for these is the set S^n of *words* or n-tuples of elements from an *alphabet* S of q symbols. The *distance* between two words is the number of coordinates in which they differ. A *code* is simply a subset of S^n. If it is to correct d errors, we must ensure that there is at most one codeword distant d or less from any given word; the triangle inequality shows that this will be achieved if the shortest distance between two codewords is at least $2d + 1$.

An important special case is that where S is a finite field GF(q), and the code C is *linear* (that is, a subspace of S^n). In this case, the distance between two codewords is just the *weight* of their difference (the number of non-zero coordinates). Thus it is important to know the *weight distribution* of a code, the number of codewords of each weight. The *MacWilliams identities* show that the weight distribution of a code C determines that of its *dual code* $C^\perp$ (with respect to the standard inner product on S^n). Using them, together with classical invariant theory, Gleason found the general form of the weight distribution of a self-dual code over GF(2). For details and generalisations, we refer to the book "Error-correcting Codes" by MacWilliams and Sloane.

Given a design on v points, each block can be represented by a v-tuple, having ones in the positions corresponding to the points of the block and zeros elsewhere. The subspace spanned by all such v-tuples over GF(q) (for some q) is a linear code. Under suitable conditions, this code, or a modification of it, is self-dual. Information about the weight distribution of the code interacts with structural information about the design. Furthermore, the permutation representation of any automorphism group of the design admits the code as an invariant subspace. Hall gives a survey of the connection between codes and designs. Highlights include the proof that a projective plane of order 10 (if any exists) can possess no collineation of order 5, and the construction of a new symmetric design on 41 points.

Two other papers also study planes of order 10. A *k-arc* in a projective plane is a set of k points no three of which are collinear; a k-arc is *complete* if it is not contained in a larger arc. Bruen relates the existence of a complete 6-arc to the existence of a set of complete 9-arcs and a set of complete 10-arcs in a plane of order 10. Coding theory methods enter into this paper, and are central to the paper of Assmus and Novillo Sardi. By considering the geometric configurations (on 16 or 20 points) defined by codewords of weight 16 or 20, they are led to consider a generalization of Steiner systems, in which blocks have 4 or 6 points and any 3 points lie in a unique block.

Finally in this area, Liebler turns his attention to projective geometry codes, where the structures in question are defined by subspaces of various dimensions in PG(3,q), and proves a new inclusion relation among the codes, using the representation theory of a suitable cyclic collineation group.

Tallini's paper is related in a different way to the problem of characterising projective geometry. So far, we have considered the points of a projective space as its basic objects. It is possible to take the set of i-dimensional subspaces for any value of i, and assign structure to it. (These sets are sometimes known as Grassman manifolds.) The Grassman manifold whose "points" are the lines of projective space has certain subsets called "lines", a Grassman "line" consisting of all the projective lines lying in a plane and passing through a point. Tallini gives axioms characterising this geometry.

We turn now to the other aspect of projective geometry, the study of properties of subsets of the point set. This can be subdivided, somewhat arbitrarily, into two parts: the geometric structure of a subset (for example, the configuration formed by the lines it contains), and the cardinalities of the intersections of lines (or other subspaces) with the subset.

Some of the most familiar subsets of projective spaces are quadrics and Hermitian varieties (the sets of zeros of non-singular quadratic or Hermitian forms). Each of these, equipped with the subspaces it contains, forms a geometry known as a (classical) *polar space*. Further polar spaces are defined by non-singular alternating bilinear forms: in this case the points of the polar space are all the points of the projective space, but only those subspaces which are totally isotropic (that is, on which the form is identically zero) belong to the polar space. These polar spaces are sometimes known by the same names as the classical groups associated with them, namely *orthogonal*, *unitary* and *symplectic*.

Classical polar spaces are given to us embedded in projective spaces. One may ask, supposing the polar spaces known, just how can they be embedded in projective spaces? Lefèvre-Percsy in her paper, surveys results on this question, and proves some new ones.

The problem of axiomatising polar spaces was solved by Tits in 1974, in his lecture notes on "Buildings of Spherical Type and Finite BN-Pairs". Tits defines an (abstract) polar space of rank n to be a set equipped with a collection of distinguished subsets called subspaces, having the following properties:

1. Any subspace, together with the subspaces it contains, is a projective space of dimension at most $n-1$.

2. Any intersection of subspaces is a subspace.
3. If M is an $(n-1)$-dimensional subspace and x a point, then the union of the set of lines joining x to points of M is an $(n-1)$-dimensional subspace.
4. There exist two disjoint $(n-1)$-dimensional subspaces.

(Tits uses the term "projective space" in a more general sense than the one we have defined. In place of our axiom 1, it is required only that each line contains at least two points. However, it is straightforward to show that the point set of such a generalised projective space is a disjoint union of point sets of restricted projective spaces, the lines being all those of the constituent spaces together with all pairs of points in different constituents.)

The result of Tits relevant to us is that a finite abstract polar space of rank at least 3, in which all lines have at least three points, is a classical polar space of one of the types described earlier.

Subsequently, Buekenhout and Shult gave a simpler axiom scheme, involving only the points and lines. Their axioms were as follows:

1. Any line has at least three points.
2. If a point x is not incident with a line L, then x is collinear with one or all points of L.
3. There exists a line; and no point is collinear with all others.

They showed that, starting from these hypotheses, it is possible to reconstruct all the subspaces and verify Tits' axioms.

It should be noticed that, as in the axiomatisation of projective spaces, the theorem only applies provided the dimension is sufficiently large: polar spaces of rank 2 are not covered by Tits' theorem. For these, the maximal subspaces are lines, and a stronger version of Buekenhout and Shult's second axiom holds: if x is not incident with L, then x is collinear with just one point of L. Such a geometry is called a *generalised quadrangle*. Thus, generalised quadrangles stand in the same relation to polar spaces as projective planes do to projective spaces. It might then be expected that the theory of generalised quadrangles would parallel that of projective planes, with the place of PG(2,q) being taken by the classical quadrangles (the classical polar spaces of rank 2). This is indeed the case. A number of characterisations of classical quadrangles by configuration theorems as properties of automorphism groups have been given.

One of the most important of these is Tits' theorem on Moufang quadrangles. These are quadrangles admitting sufficiently many "root automorphisms" (analogous to central collineations of projective planes), and Tits has shown, in particular, that finite Moufang quadrangles are classical. (This should be seen as the analogue of the characterisation of Desarguesian projective planes mentioned earlier.) Payne and Thas, in their paper in this volume, give an alternative, more elementary, approach to this important theorem. The basic idea is to establish relations between root automorphisms and structural properties, similar to the well-known relation between central collineations and Desargues' theorem for planes.

These classes of geometries have been generalised further. It is possible to define *generalised n-gons* for any $n \geq 2$. The precise definition does not concern us here; it suffices to say that projective planes are generalised 3-gons, and just as we saw that a projective plane gives rise to an extremal graph of girth 6 , so a generalised n-gon gives an extremal graph of girth 2n . Generalised polygons are the objects from which *buildings* are constructed, in much the same way as polytopes (or Coxeter complexes) are built from ordinary polygons. The class of buildings includes projective and polar spaces. Buildings are associated with the finite simple groups of "Lie type".

A more general class of *diagram geometries* has been defined by Buekenhout. He allows a wider class of "building blocks", and a more general "construction method". As a result, he finds geometries associated with many of the recently-discovered "sporadic" simple groups. A diagram geometry (in Buekenhout's sense) which is built from generalised polygons is called a *Tits geometry* by Ott and Ronan, who continue the programme (initiated by Tits) of studying such geometries. Much of this study has a topological flavour. Any "geometry" which consists of subspaces of various dimensions or "types" can be described as a simplicial complex, where a simplex is a flag or set of mutually incident subspaces. (Our graph of girth 6 constructed from a projective plane earlier is an example.) Ott and Ronan study the universal covers of Tits geometries, showing that under certain conditions they are necessarily buildings. Ronan examines similar problems of universal covers for some sporadic Buekenhout geometries.

Another method, that of Hecke algebras or incidence algebras, underlies the celebrated result of Feit and Higman : this asserts that a finite generalised n-gon with $s+1$ points on each line and $t+1$ lines through each point $(s, t > 1)$ can exist only if $n = 2,3,4, 6$ or 8. Ott extends this method to wider classes of diagram geometries.

The article by Hall and Shult may be fitted loosely into this framework. Consider a polar space with three points on every line. (Such a space is necessarily of symplectic or orthogonal type over GF(2) : there are no non-classical quadrangles with $s = 2$.) Form a graph, whose vertices are the points of the polar space, two vertices adjacent whenever they are collinear. Examining the Buekenhout-Shult axioms, we see that such graphs are characterised by the following *triangle property :*

> (*) any edge xy lies in a triangle xyz having the property that any further vertex is joined to one or all of x, y and z ;

together with the nondegeneracy conditions that there is at least one edge and that no vertex is joined to all others. Indeed, this characterisation, given by Shult and Seidel, preceded and motivated the Buekenhout-Shult theorem. The same authors considered a variation, the *cotriangle property*, obtained from (*) by replacing "edge" and "triangle" by "non-edge" and "cotriangle" (a cotriangle being a set of three pairwise nonadjacent vertices).