Applied Analysis

Mathematical Methods in Natural Science

Takasi Senba · Takashi Suzuki

Imperial College Press

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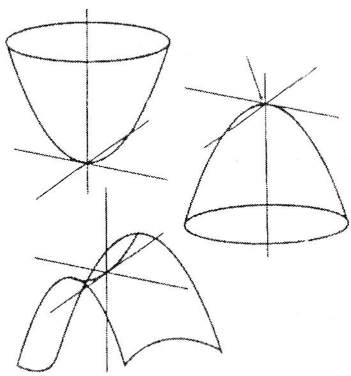

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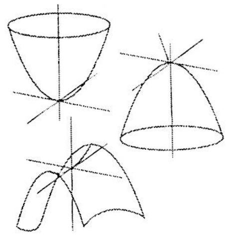

Dedicated to Professor Hiroshi Fujita

Preface

This book is intended to be an introduction to mathematical science, particularly the theoretical study from the viewpoint of applied analysis. As basic materials, *vector analysis* and *calculus of variation* are taken, and then *Fourier analysis* is introduced for the eigenfunction expansion to justify. After that, *statistical method* is presented to control the mean field of many particles, and the mathematical theory to linear and nonlinear partial differential equations is accessed. *System of chemotaxis* is a special topic in this book, and well-posedness of the model is established. We summarize several mathematical theories and give some references for the advanced study. We also picked up some materials from classical mechanics, geometry, mathematical programming, numerical schemes, and so forth. Thus, this book covers some parts of undergraduate courses for mathematical study. It is also suitable for the first degree of graduate course to learn the basic ideas, mathematical techniques, systematic logic, physical and biological motivations, and so forth.

Most part of this monograph is based on the notes of the second author for undergraduate and graduate courses and seminars at several universities. We thank all our students for taking part in the project.

December 2003

Takasi Senba and Takashi Suzuki

Contents

Chapter 1

Geometric Objects

Some kind of insects and amoeba are lured by special chemical substances of their own. Such a character is called chemotaxis in biology. For its formulation, some mathematical terminologies and notions are necessary. This chapter is devoted to geometric objects.

1.1 Basic Notions of Vector Analysis

1.1.1 *Dynamical Systems*

Movement of a *mass point* is indicated by the position vector $\boldsymbol{x} = \boldsymbol{x}(t) \in \mathbf{R}^3$ depending on the time variable $t \in \mathbf{R}$. If m and $\boldsymbol{F}$ denote its mass and the force acting on it, respectively, *Newton's equation of motion* assures the relation

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F}, \tag{1.1}$$

where $\frac{d^2\boldsymbol{x}}{dt^2}$ stands for the *acceleration vector*. If n points $x_i = x_i(t)$ $(i = 1, 2, \cdots, n)$ are interacting, then they are subject to the system

$$m_i\frac{d^2\boldsymbol{x}_i}{dt^2} = \boldsymbol{F}_i \qquad (i = 1, 2, \cdots, n),$$

simply written as

$$\ddot{x} = f(x, \dot{x}, t) \tag{1.2}$$

with $x = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n) \in \mathbf{R}^{3n}$,

$$\ddot{x} = \frac{d^2x}{dt^2}, \qquad \text{and} \qquad \dot{x} = \frac{dx}{dt}.$$

It is sometimes referred to as the *deterministic principle* of Newton, and under reasonable assumptions on f, say, continuity in all variables $(x, \dot{x}, t)$ and the Lipschitz continuity in $(x, \dot{x})$, there is a unique solution $x = x(t)$ to (1.2) locally in time with the prescribed initial position $x(0) = x_0$ and the initial velocity $\dot{x}(0) = \dot{x}_0$. At this occasion, let us recall that the initial values

$$x(0) = x_0 \qquad \text{and} \qquad \dot{x}(0) = \dot{x}_0$$

provide equation (1.2) with the *Cauchy problem.*

In some cases the degree of freedom is reduced, as $x = x(t) \in \mathbf{R}$ or $x = x(t) \in \mathbf{R}^2$. For example,

$$\ddot{x} = -k^2 x$$

with $x = x(t) \in \mathbf{R}$ is associated with the oscillatory motion of a bullet hanged by spring, and its solution is given by

$$x(t) = x_0 \cos kt + \dot{x}_0 \sin kt/k.$$

Although very few solutions to (1.2) are written explicitly even for the case of $x = x(t) \in \mathbf{R}$,

$$\ddot{x} = f(x) \tag{1.3}$$

is the simplest but general form of it. In this case

$$T = \frac{1}{2}\dot{x}^2 \qquad \text{and} \qquad U(x) = -\int^x f(\xi)d\xi$$

are referred to as the *kinetic energy* and the *potential energy*, respectively. Then, the *total energy* is given by

$$E = T + U = \frac{1}{2}\dot{x}^2 + U(x)$$

so that it is a function of $(x, \dot{x})$, denoted by $E = E(x, \dot{x})$. If $x = x(t)$ is a solution to (1.3), then it holds that

$$\frac{d}{dt}E\left(x(t), \dot{x}(t)\right) = \dot{x}\ddot{x} - f(x)\dot{x} = \dot{x}\left(\ddot{x} - f(x)\right) = 0,$$

so that $E(x(t), \dot{x}(t))$ is a constant. This fact is referred to as the *conservation law of energy.*

System (1.3) is equivalent to $\dot{x} = y$ and $\dot{y} = f(x)$, or

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(x, y) \tag{1.4}$$

with $\Phi(x, y) = {}^t(y, f(x))$. Because the right-hand side does not include the variable t explicitly, system (1.4) is said to be *autonomous*, and its solution is illustrated as a curve in $x - y$ plane. Energy conservation $E(x(t), \dot{x}(t)) = E(x_0, \dot{x}_0)$ guarantees the existence of the solution globally in time if the potential $U = U(x)$ is *coercive*, which means that $|x| \to +\infty$ implies $U(x) \to +\infty$. Then, each

$$\mathcal{O} = \{(x(t), \dot{x}(t)) \mid t \in \mathbf{R}\}$$

is called an *orbit*, which coincides with the curve $E = \frac{1}{2}y^2 + U(x)$, where $E = E(x_0, \dot{x}_0)$.

Because of the uniqueness of the solution to the Cauchy problem of (1.3), the orbit never intersects by itself. However, it may be a point, which corresponds to the zero of Φ, that is, $y = 0$ and $f(x) = -U'(x) = 0$. It is referred to as the *equilibrium point.* Each equilibrium point $(\hat{x}_0, 0)$ is *stable* or *unstable* if $\hat{x}_0$ is a local minimum or a local maximum of $U = U(x)$, respectively. This means that if the initial value $(x_0, \dot{x}_0)$ is close to $(\hat{x}_0, 0)$, then the solution to (1.3) stays near or away from it.

The solution to (1.4) may be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = T_t \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

for ${}^t(x(0), y(0)) = {}^t(x_0, \dot{x}_0)$, with the mapping $T_t : \mathbf{R}^2 \to \mathbf{R}^2$ defined for each $t \in \mathbf{R}$. Then the family $\{T_t\}_{t \in \mathbf{R}}$ induces the continuous mapping

$$T : \mathbf{R}^2 \times \mathbf{R} \quad \to \quad \mathbf{R}^2$$

by

$$T\left({}^t(x, y), t\right) = T_t \begin{pmatrix} x \\ y \end{pmatrix}.$$

This family is provided with the properties that $T_0 = Id$, the identity operator, and $T_{t+s} = T_t \circ T_s$ for $t, s \in \mathbf{R}$, with $\circ$ denoting the composition of operators. Then, we call $\{T_t\}_{t\in\mathbf{R}}$ the *dynamical system.*

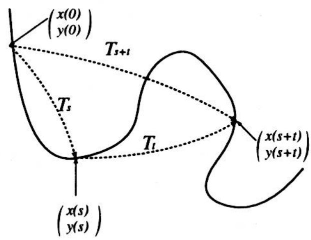

Fig. 1.1

For $x = x(t) \in \mathbf{R}^3$ in (1.3), we say that $f = f(x)$ is a *potential field* if

$$f = -\begin{pmatrix} \partial U/\partial x_1 \\ \partial U/\partial x_2 \\ \partial U/\partial x_3 \end{pmatrix} \tag{1.5}$$

holds with a scalar function $U = U(x_1, x_2, x_3)$. In use of the *gradient operator*

$$\nabla = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix},$$

relation (1.5) is written as

$$f = -\nabla U.$$

Then we can define the total energy by

$$E(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 + U(x),$$

where $|\dot{x}|$ denotes the length of the velocity vector $\dot{x} \in \mathbf{R}^3$. Similarly to the one-dimensional case, this E is a quantity of conservation. In fact, if $x = x(t)$ is a solution to (1.3) it follows that

$$\frac{d}{dt}E\left(x(t), \dot{x}(t)\right) = \dot{x} \cdot (\ddot{x} + \nabla U(x)) = 0,$$

where $\cdot$ denotes the inner product in $\mathbf{R}^3$. However, this law of the conservation of energy is not sufficient to control the orbit $\mathcal{O}$ in $x - \dot{x}$ space, which is now identified with $\mathbf{R}^6$.

Exercise 1.1 Illustrate some orbits to (1.3) for $x = x(t) \in \mathbf{R}$ in $x - \dot{x}$ plane, when the potential energy is given by $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. Seek all equilibrium points and judge their stability. Examine the same question for $U(x) = \pm\frac{1}{2}x^2$.

1.1.2 *Outer Product*

Here, we take the notion of vector analysis; outer product of the vector, gradient of the scalar field, and divergence and rotation of the vector field. Throughout the present chapter, three-dimensional vectors are denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\cdots$, while a, b, c, $\cdots$ indicate scalars. The canonical basis of $\mathbf{R}^3$ is given by

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which are arranged to form a right-handed coordinate system in three dimensional space $\mathbf{R}^3$. The length of $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$, and $\boldsymbol{a} \cdot \boldsymbol{b}$ stands for the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$. That is, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. If

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \tag{1.6}$$

and

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} \tag{1.7}$$

represent those vectors by their components, it holds that $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The outer product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector denoted by $\boldsymbol{a} \times \boldsymbol{b}$ satisfying the following property.

1 Its length is equal to the area of the parallelogram made by $\boldsymbol{a}$ and $\boldsymbol{b}$.
2 It is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.
3 $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{a} \times \boldsymbol{b}$ are right-handed.

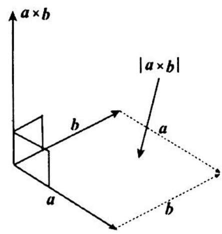

Fig. 1.2

Then, we have the following.

Theorem 1.1 *The operation* $(\boldsymbol{a}, \boldsymbol{b}) \mapsto \boldsymbol{a} \times \boldsymbol{b}$ *is subject to the following laws.*

1 *(commutative):* $\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$.
2 *(associative):* $c(\boldsymbol{a} \times \boldsymbol{b}) = (c\boldsymbol{a}) \times \boldsymbol{b}$.
3 *(distributive):* $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

Proof. We shall show the distributive law because the other laws are obvious. First, from the associative law we may suppose that $|\boldsymbol{a}| = 1$. We take the plane π containing the origin whose normal vector is $\boldsymbol{a}$. Look down π so that $\boldsymbol{a}$ is upward. Let $\boldsymbol{b}'$, $\boldsymbol{c}'$, and $(\boldsymbol{b} + \boldsymbol{c})'$ be the projections to π of $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{b} + \boldsymbol{c}$, respectively. Then, by the definition we have

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}', \qquad \boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c}',$$

and

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})'.$$

Here, we have $(\boldsymbol{b} + \boldsymbol{c})' = \boldsymbol{b}' + \boldsymbol{c}'$, so that the equality

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c},$$