DIFFERENCE EQUATIONS

An Introduction with Applications

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To our wives, Marilyn and Tina

Preface

This book uses elementary analysis and linear algebra to investigate solutions to difference equations. We expect that the reader will have encountered difference equations in one or more of the following contexts: the approximation of solutions of equations by Newton's method, the discretization of differential equations, the computation of special functions, the counting of elements in a defined set (combinatorics), and the discrete modelling of economic or biological phenomena. In this book, we give examples of how difference equations arise in each of these areas, as well as examples of numerous applications to other subjects.

Our goal is to present an overview of the various facets of difference equations that can be studied by elementary mathematical methods. We hope to convince the reader that difference equations is a rich field, both interesting and useful. The reader will not find here a text on numerical analysis (plenty of good ones already exist). Although much of the contents of this book is closely related to the techniques of numerical analysis, we have, except in a few places, omitted discussion of the concerns of computation by computer.

We assume that the reader has no background in difference equations. The first three chapters provide an elementary introduction to the subject. A good course in calculus should suffice as a preliminary to reading this material. Chapter 1 gives seven elementary examples, including the definition of the Gamma function, which will be important in later chapters. Chapter 2 surveys briefly the fundamentals of difference calculus. In Chapter 3, the basic theory for linear difference equations is developed, and several methods are given for finding closed form solutions, including annihilators, generating functions and z-transforms. Also included are sections on applications and on transforming nonlinear equations into linear ones.

Chapter 4, which is essentially independent from the earlier chapters, is concerned mainly with stability theory for autonomous systems of equations. The Putzer algorithm for computing A^t, where A is an n by n matrix, is presented, leading to the solution of autonomous linear systems with constant coefficients. The chapter covers many of the fundamental stability results for linear and nonlinear systems, using eigenvalue criteria, stairstep diagrams, Liapunov functions and linearization. The last section is a brief introduction to chaotic behavior.

Approximations of solutions to difference equations for large values of the independent variable are studied in Chapter 5. This chapter is mostly independent of Chapter 4, but does use some of the results from Chapters 2 and 3. Here, one will find the asymptotic analysis of sums, the theorems of Poincaré and Perron on asymptotic behavior of solutions to linear equations, and the asymptotic behavior of solutions to nonlinear autonomous equations, with applications to Newton's method and the modified Newton's method.

Chapters 6 through 9 develop a wide variety of distinct but related topics involving second order difference equations from the theory given in Chapter 3. Chapter 6 contains a detailed study of the self-adjoint equation. This chapter includes generalized zeros, interlacing of zeros of independent solutions, disconjugacy, Green's functions, boundary value problems for linear equations, Riccati equations, and oscillation of solutions. Sturm-Liouville problems for difference equations are considered in Chapter 7. These problems lead to a consideration of finite Fourier series, properties of eigenpairs for self-adjoint Sturm-Liouville problems, nonhomogeneous problems, and a Rayleigh inequality for finding upper bounds on the smallest eigenvalue. Chapter 8 treats the discrete calculus of variations for sums, including the Euler-Lagrange difference equation, transversality conditions, the Legendre necessary condition for a local extremum, and some sufficient conditions. Disconjugacy plays an important role here and, indeed, the methods in this chapter are used to sharpen some of the results from Chapter 6. In Chapter 9, several existence and uniqueness results for nonlinear boundary value problems are proved, using the contraction mapping theorem and Brouwer fixed point theorems in Euclidean space. A final section relates these results to similar theorems for differential equations.

The last chapter takes a brief look at partial difference equations. It is shown how these arise from the discretization of partial differential equations. Computational molecules are introduced in order to determine what sort of initial and boundary conditions are needed to produce unique solutions of partial difference equations. Some special methods for finding explicit solutions are summarized.

This book can be used as a textbook at a variety of different levels ranging from middle undergraduate to beginning graduate, depending on the choice of topics. There are many exercises of varying degrees of difficulty (120 just in Chapter 3 alone). Answers to selected problems can be found near the end of the book. There is also a large bibliography of books and papers on difference equations for further study.

Preliminary portions of this book have been used by the authors in courses at the University of Oklahoma and the University of Nebraska. We are indebted to D. Hankerson and J. Hooker, who have taught courses from segments of the book at Auburn University and Southern Illinois University at Carbondale, respectively, and have offered helpful suggestions. We would also like to thank the following individuals who have influenced the book directly or indirectly: C. Ahlbrandt, G. Diaz, S. Elaydi, P. Eloe, L. Erbe, B. Harris, J. Henderson, L. Hall, L. Jackson, G. Ladas, R. Nau, W. Patula, T. Peil, J. Ridenhour, J. Schneider, and D. Smith. Shireen Ray deserves a special word of thanks for expertly "TEX-ing" the manuscript. Finally, Walter Kelley is grateful to the Graduate College and the Research Council at the University of Oklahoma for travel support during the writing of the book.

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Chapter 1
Introduction

Mathematical computations frequently are based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called a "difference equation" or "recurrence equation." These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields.

The following elementary examples have been chosen to illustrate something of the diversity of the uses of difference equations and of the types of these equations that arise. Many more examples will appear later in the book.

Example 1.1. In 1626, Peter Minuit purchased Manhattan Island for goods worth \$24. If the \$24 could have been invested at an annual interest rate of 7% compounded quarterly, what would it have been worth in 1986?

Let $y(t)$ be the value of the investment after t quarters of a year. Then $y(0) = 24$. Since the interest rate is 1.75% per quarter, $y(t)$ satisfies the difference equation

$$\begin{aligned} y(t+1) &= y(t) + .0175y(t) \\ &= (1.0175)y(t) \end{aligned}$$

for $t = 0, 1, 2, \cdots$. Computing y recursively, we have

$$\begin{aligned} y(1) &= 24(1.0175), \\ y(2) &= 24(1.0175)^2, \\ &\vdots \\ y(t) &= 24(1.0175)^t. \end{aligned}$$

After 360 years, or 1440 quarters, the value of the investment is

$$\begin{aligned} y(1440) &= 24(1.0175)^{1440} \\ &\simeq 1.697 \times 10^{12} \end{aligned}$$

(about 1.7 trillion dollars!).

Example 1.2. It is observed that the decrease in the mass of a radioactive substance over a fixed time period is proportional to the mass that was present at the beginning of the time period. If the half life of radium is 1600 years, find a formula for its mass as a function of time.

Let $m(t)$ represent the mass of the radium after t years. Then

$$m(t+1) - m(t) = -km(t),$$

where k is a positive constant. Then

$$m(t+1) = (1-k)m(t)$$

for $t = 0, 1, 2, \cdots$. Using iteration as in the preceding example, we find

$$m(t) = m(0)(1-k)^t.$$

Since the half life is 1600,

$$m(1600) = m(0)(1-k)^{1600} = \frac{1}{2}m(0),$$

so

$$1 - k = \left(\frac{1}{2}\right)^{\frac{1}{1600}},$$

and we have finally that

$$m(t) = m(0)\left(\frac{1}{2}\right)^{\frac{t}{1600}}.$$

This problem is traditionally solved in calculus and physics textbooks by setting up and integrating the differential equation $m'(t) = -km(t)$. However, the solution presented here using a difference equation is somewhat shorter and employs only elementary algebra.

Example 1.3. (The Tower of Hanoi Problem) The problem is to find the minimum number of moves $y(t)$ required to move t rings from the first peg in Figure 1.1 to the third peg. A move consists of transferring a single ring from one peg to another with the restriction that a larger ring may not be placed on a smaller ring. The reader should find $y(t)$ for some small values of t before reading further.

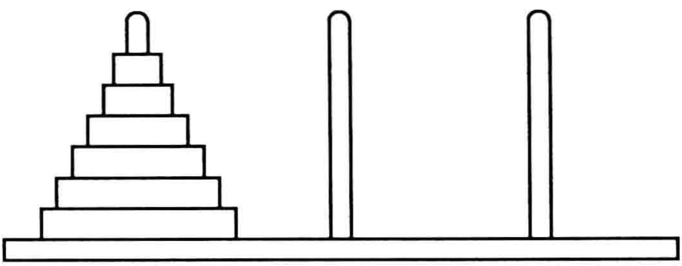

Fig. 1.1 Initial position of the rings

We can find the solution of this problem by finding a relationship between $y(t+1)$ and $y(t)$. Suppose there are $t+1$ rings to be moved. An essential intermediate stage in a successful solution is shown in Fig. 1.2. Note that exactly $y(t)$ moves are required to obtain this arrangement since the minimum number of moves needed to move t rings from peg 1 to peg 2 is the same as the minimum number of moves to move t rings from peg 1 to peg 3. Now a single move places the largest ring on peg 3, and $y(t)$ additional moves are needed to move the other t rings from peg 2 to peg 3. We are led to the difference equation

$$y(t+1) = y(t) + 1 + y(t),$$

or

$$y(t+1) - 2y(t) = 1.$$

The solution which satisfies $y(1) = 1$ is

$$y(t) = 2^t - 1.$$

(See Exercise 1.7.) Check the answers you got for $t = 2$ and $t = 3$.

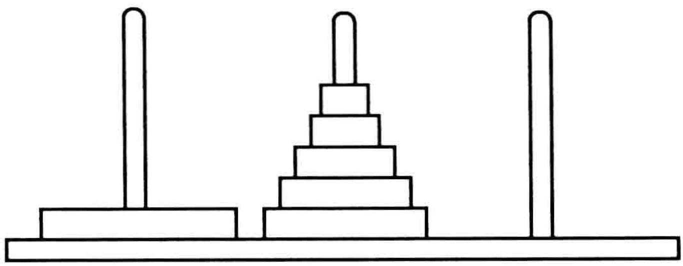

Fig. 1.2 An intermediate position

Example 1.4. (Airy equation) Suppose we wish to solve the differential equation

$$y''(x) = x\ y(x).$$

The Airy equation appears in many calculations in applied mathematics, e.g., in the study of nearly discontinuous periodic flow of electric current and in the description of the motion of particles governed by the Schrodinger equation in quantum mechanics. One approach is to seek power series solutions of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Substitution of the series into the differential equation yields

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} = \sum_{k=0}^{\infty} a_k x^{k+1}.$$

The change of index $k \to k+3$ in the series on the left side of the equation gives us

$$\sum_{k=-1}^{\infty} a_{k+3}(k+3)(k+2)x^{k+1} = \sum_{k=0}^{\infty} a_k x^{k+1}.$$

In order that these series be equal for an interval of x values, the coefficients of x^{k+1} must be the same for all $k = -1, 0, \cdots$. For $k = -1$, we have

$$a_2(2)(1) = 0,$$

so $a_2 = 0$. For $k = 0, 1, 2, \cdots$,

$$a_{k+3}(k+3)(k+2) = a_k$$

or

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}.$$

The last equation is a difference equation that allows us to compute (in principle) all coefficients a_k in terms of the coefficients a_0 and a_1. Note that $a_{3n+2} = 0$ for $n = 0, 1, 2, \cdots$ since $a_2 = 0$.

Treating a_0 and a_1 as arbitrary constants we obtain the general solution of the Airy equation expressed as a power series:

$$y(x) = a_0\left[1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right] + a_1\left[x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right].$$

Returning to the difference equation, we have

$$\frac{a_{k+3}}{a_k} = \frac{1}{(k+3)(k+2)} \to 0 \quad \text{as } k \to \infty,$$

and the ratio test implies that the power series converges for all values of x.

Example 1.5. Suppose a sack contains r red marbles and g green marbles. The following procedure is repeated n times: a marble is drawn at random from the sack, its color is noted and it is replaced. We want to compute the number $W(n, k)$ of ways of obtaining exactly k red marbles among the n draws.

We will be taking the order in which marbles are drawn into account here. For example, if the sack contains two red marbles R_1, R_2 and one green marble G, then the possible outcomes with $n = 2$ draws are GG, GR_1, GR_2, R_1R_1, R_1R_2, R_1G, R_2R_1, R_2R_2 and R_2G, so $W(2,0) = 1, W(2,1) = 4$ and $W(2,2) = 4$.

There are two cases. In the first case, the k^{th} red marble is drawn on the n^{th} draw. Since there are $W(n-1, k-1)$ ways of drawing $k-1$ red marbles on the first $n-1$ draws, the total number of ways that this case can occur is $rW(n-1, k-1)$.

In the second case, a green marble is drawn on the n^{th} draw. The k red marbles were drawn on the first $n-1$ draws, so in this case the total is $gW(n-1, k)$.

Since these two cases are exhaustive and mutually exclusive, we have

$$W(n, k) = rW(n-1, k-1) + gW(n-1, k),$$

which is a difference equation in two variables, sometimes called a "partial difference equation." Mathematical induction can be used to verify the formula

$$W(n, k) = \binom{n}{k} r^k g^{n-k},$$

where $k = 0, 1, \cdots, n$ and $n = 1, 2, 3, \cdots$. The notation $\binom{n}{k}$ represents the binomial coefficient $\frac{n!}{k!(n-k)!}$.

From the Binomial Theorem, the total number of possible outcomes is

$$\sum_{k=0}^{n} \binom{n}{k} r^k g^{n-k} = (r+g)^n,$$

so the probability of drawing exactly k red marbles is

$$\frac{\binom{n}{k} r^k g^{n-k}}{(r+g)^n} = \binom{n}{k} \left(\frac{r}{r+g}\right)^k \left(\frac{g}{r+g}\right)^{n-k},$$

a fundamental formula in probability theory.

Example 1.6. Perhaps the most useful of the higher transcendental functions is the gamma function $\Gamma(z)$, which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

if the real part of z is positive. Formally applying integration by parts, we have

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t} t^z dt \\ &= [-e^{-t} t^z]_o^\infty - \int_0^\infty (-e^{-t}) z\, t^{z-1} dt \\ &= z \int_0^\infty e^{-t} t^{z-1} dt,\end{aligned}$$

so that Γ satisfies the difference equation

$$\Gamma(z+1) = z\Gamma(z).$$

Note that here, as in Example 1.2, the independent variable is not restricted to discrete values. If the value of $\Gamma(z)$ is known for some z whose real part belongs to $(0,1)$, then we can compute $\Gamma(z+1), \Gamma(z+2), \cdots$ recursively. Furthermore, if we write the difference equation in the form

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

then $\Gamma(z)$ can be given a useful meaning for all z with the real part less than or equal to zero except $z = 0, -1, -2, \cdots$.