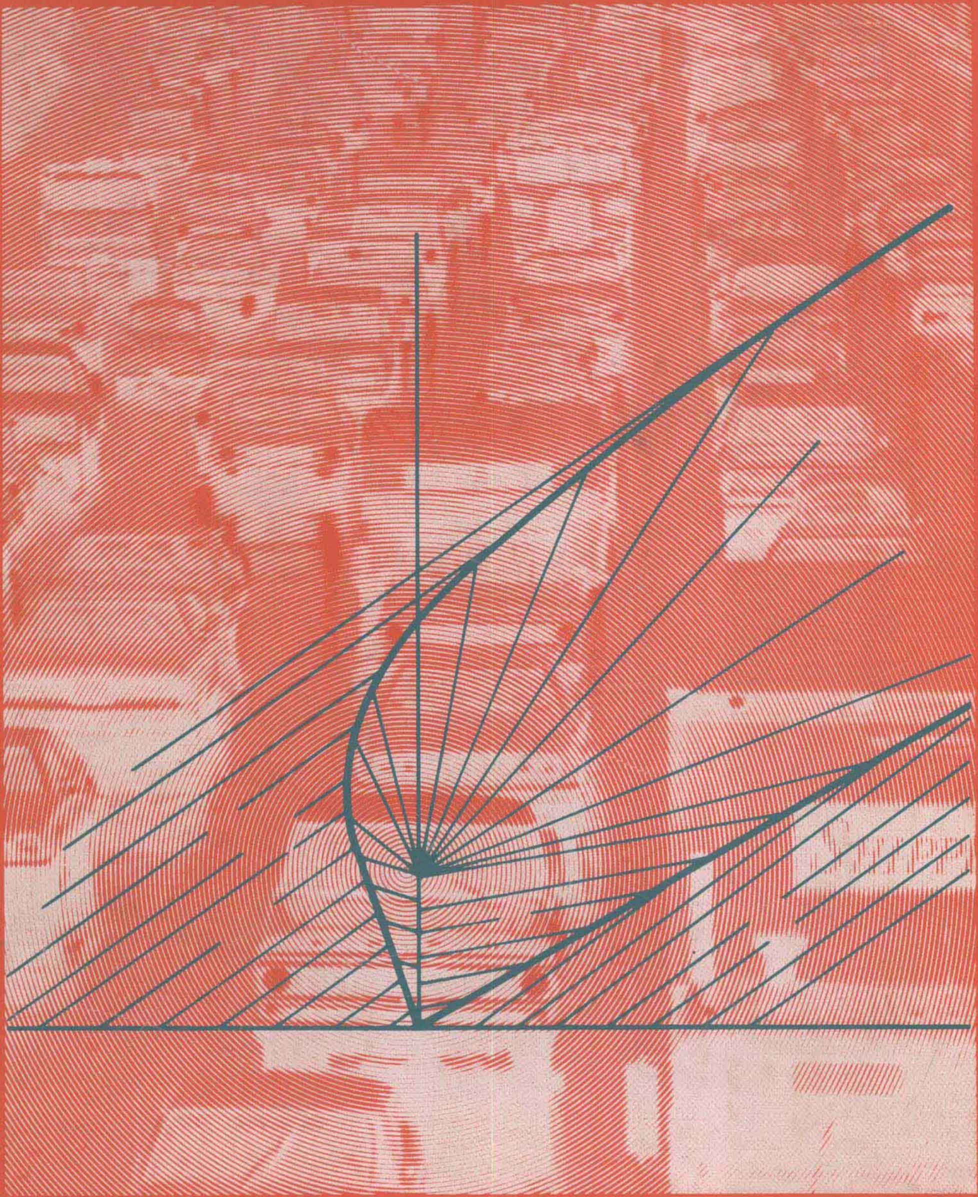
Mathematical Models

Mechanical Vibrations, Population Dynamics, and Traffic Flow



RICHARD HABERMAN

Mathematical Models

Mechanical Vibrations, Population Dynamics, and Traffic Flow

(AN INTRODUCTION TO APPLIED MATHEMATICS)

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Mathematical Models

Preface

I believe the primary reason for studying mathematics lies in its applications. By studying three diverse areas in which mathematics has been applied, this text attempts to introduce to the reader some of the fundamental concepts and techniques of applied mathematics. In each area, relevant observations and experiments are discussed. In this way a mathematical model is carefully formulated. The resulting mathematical problem is solved, requiring at times the introduction of new mathematical methods. The solution is then interpreted, and the validity of the mathematical model is questioned. Often the mathematical model must be modified and the process of formulation, solution, and interpretation continued. Thus we will be illustrating the relationships between each area and the appropriate mathematics. Since one area at a time is investigated in depth in this way, the reader has the opportunity to understand each topic, not just the mathematical techniques.

Mechanical vibrations, population dynamics, and traffic flow are chosen as areas to investigate in an introduction to applied mathematics for similar reasons. In each, the experiments and common observations necessary to formulate and understand the mathematical models are relatively well known to the average reader. We will not find it necessary to refer to exceedingly technical research results. Furthermore these three topics were chosen for inclusion in this text because each serves as an introduction to more specialized investigations. Here we attempt only to introduce these various topics and leave the reader to pursue those of most interest. Mechanical vibrations (more specifically the motion of spring-mass systems and pendulums) is naturally followed by a study of other topics from physics; mathematical ecology (involving the population growth of species interacting with their

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environment) is a possible first topic in biomathematics; and traffic flow (investigating the fluctuations of traffic density along a highway) introduces the reader in a simpler context to many mathematical and physical concepts common in various areas of engineering, such as heat transfer and fluid dynamics. In addition, it is hoped that the reader will find these three topics as interesting as the author does.

A previous exposure to physics will aid the reader in the part on mechanical vibrations, but the text is readily accessible to those without this background. The topics discussed supplement rather than substitute for an introductory physics course. The material on population dynamics requires no background in biology; experimental motivation is self-contained. Similarly, there is sufficient familarity with traffic situations to enable the reader to thoroughly understand the traffic models that are developed.

This text has been written with the assumption that the reader has had the equivalent of the usual first two years of college mathematics (calculus and some elementary ordinary differential equations). Many critical aspects of these prerequisites are briefly reviewed. More specifically, a knowledge of calculus including partial derivatives is required, but vector integral calculus (for example, the divergence theorem) is never used (nor is it needed). Linear algebra and probability are also not required (although they are briefly utilized in a few sections which the reader may skip). Although some knowledge of differential equations is required, it is mostly restricted to first and second order constant coefficient equations. A background in more advanced techniques is not necessary, as they are fully explained where needed.

Mathematically, the discussion of mechanical vibrations and population dynamics proceed in similar ways. In both, emphasis is placed on the nonlinear aspects of ordinary differential equations. The concepts of equilibrium solutions and their stability are developed, considered by many to be one of the fundamental unifying themes of applied mathematics. Phase plane methods are introduced and linearization procedures are explained in both parts. On the other hand, the mathematical models of traffic flow involve first-order (nonlinear) partial differential equations, and hence is relatively independent of the previous material. The method of characteristics is slowly and carefully explained, resulting in the concept of traffic density wave propagation. Throughout, mathematical techniques are developed, but equal emphasis is placed on the mathematical formulation of the problem and the interpretation of the results.

I believe, in order to learn mathematics, the reader must take an active part. This is best accomplished by attempting a significant number of the included exercises. Many more problems are included than are reasonable for the average reader to do. The exercises have been designed such that their difficulty varies. Almost all readers will probably find some too easy, while others are quite difficult. Most are word problems, enabling the reader to consider the relationships between the mathematics and the models.

Each major part is divided into many subsections. However, these sections are not of equal length. Few correspond to as much as a single lecture. Usually more than one (and occasionally, depending on the background of the reader, many) of the sections can be covered in an amount of time equal to that of a single lecture. In this way the book has been designed to be substantially covered in one semester. However, a longer treatment of these subjects will be beneficial for some. Furthermore, with material added by individual instructors, this text may be used as the basis of a full year's introduction to applied mathematics. For others, a second semester of applied mathematics could consist of, for example, the heat, wave and Laplace's equation (and the mathematics of Fourier series as motivated by separation of variables of these partial differential equations).

This text is a reflection of my own philosophy of applied mathematics. However, anyone's philosophy is strongly influenced by one's exposure. For my own education, the applied mathematics group at the Massachusetts Institute of Technology must be sincerely thanked. Any credit for much of this book must be shared with them in some ill-defined way.

A course has been offered for a few years based on preliminary versions of this text. Student comments have been most helpful as have been the insights given to me by Dr. Eugene Speer and Dr. Richard Falk who have co-taught the material with me. Also I would like to express my appreciation to Dr. Mark Ablowitz for his many thoughtful and useful suggestions.

For the opportunity and encouragement to develop an applied mathematics course for which this text was written, I wish sincerely to thank Dr. Terry Butler. Furthermore his interest in the needs of students reinforced my own attitudes and resulted in this text.

Besides the usual gratitude to one's wife, my thanks to Liz for the thankless task of helping in rewriting the many drafts. Having no interest or knowledge in mathematics, this was an exceptionally difficult effort.

My appreciation to the typists of the manuscript (originally class notes), especially Mrs. Annette Roselli whose accurate work was second only to her patience with the numerous revisions.

RICHARD HABERMAN

Contents

(Starred sections may be omitted without loss of continuity.)

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Mechanical Vibrations

1. Introduction to Mathematical Models in the Physical Sciences

Science attempts to establish an understanding of all types of phenomena. Many different explanations can sometimes be given that agree qualitatively with experiments or observations. However, when theory and experiment quantitatively agree, then we can usually be more confident in the validity of the theory. In this manner mathematics becomes an integral part of the scientific method.

Applied mathematics can be said to involve three steps:

1. the formulation of a problem—the approximations and assumptions, based on experiments or observations, that are necessary to develop, simplify, and understand the mathematical model;
2. the solving of realistic problems (including relevant computations);
3. the interpretation of the mathematical results in the context of the nonmathematical problem.

In this text, we will attempt to give equal emphasis to all three aspects.

One cannot underestimate the importance of good experiments in developing mathematical models. However, mathematical models are important in their own right, aside from an attempt to mimic nature. This occurs because the real world consists of many interacting processes. It may be impossible in an experiment to entirely eliminate certain undesirable effects. Furthermore one is never sure which effects may be negligible in nature. A mathematical model has an advantage in that we are able to consider only certain effects, the object being to see which effects account for given observations and which effects are immaterial.

The process of applying mathematics never ends. As new experiments or observations are made, the mathematical model is continually revised and improved. To illustrate this we first study some problems from physics involving mechanical vibrations.

A spring-mass system is analyzed, simplified by many approximations including linearization (Secs. 2–9). Experimental observations necessitate the consideration of frictional forces (Secs. 10–13). A pendulum is then analyzed (Secs. 14–16) since its properties are similar to those of a spring-mass system. The nonlinear frictionless pendulum and spring-mass systems are briefly studied, stressing the concepts of equilibrium and stability (Secs. 17–18), before energy principles and phase plane analysis are used (Secs. 19–20). Examples of nonlinear frictionless oscillators are worked out in detail (Secs. 21–25). Nonlinear systems which are damped are then discussed (Secs. 26–28). Mathematical models of increasing difficulty are formulated; we proceed in the following manner:

1. linear systems (frictionless).
2. linear systems with friction.
3. nonlinear systems (frictionless).
4. nonlinear systems with friction.

2. Newton's Law

To begin our investigations of mathematical models, a problem with which most of you are somewhat familiar will be considered. We will discuss the motion of a mass attached to a spring as shown in Fig. 2-1:

Figure 2-1 Spring-mass system.

Observations of this kind of apparatus show that the mass, once set in motion, moves back and forth (oscillates). Although few people today have any intrinsic interest in such a spring-mass system, historically this problem played an important part in the development of physics. Furthermore, this simple spring-mass system exhibits behavior of more complex systems. For example, the oscillations of a spring-mass system resemble the motions of clock-like mechanisms and, in a sense, also aid in the understanding of the up-and-down motion of the ocean surface.

Physical problems cannot be analyzed by mathematics alone. This should be the first fundamental principle of an applied mathematician (although apparently some mathematicians would frequently wish it were not so). A spring-mass system cannot be solved without formulating an equation which describes its motion. Fortunately many experimental observations culminated in **Newton's second law of motion** describing how a particle reacts to a force. Newton discovered that the motion of a point mass is well described by the now famous formula

$$\vec{F} = \frac{d}{dt}(m\vec{v}), \tag{2.1}$$

where $\vec{F}$ is the vector sum of all forces applied to a point mass of mass m. The forces $\vec{F}$ equal the rate of change of the **momentum** $m\vec{v}$, where $\vec{v}$ is the velocity of the mass and $\vec{x}$ its position:

$$\vec{v} = \frac{d\vec{x}}{dt}. \tag{2.2}$$

If the mass is constant (which we assume throughout this text), then

$$\boxed{\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a},} \tag{2.3}$$

where $\vec{a}$ is the vector acceleration of the mass

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}. \tag{2.4}$$

Newton's second law of motion (often referred to as just **Newton's law**), equation 2.3, states that the force on a particle equals its mass times its acceleration, easily remembered as "F equals ma." The resulting acceleration of a point mass is proportional to the total force acting on the mass.

At least two assumptions are necessary for the validity of Newton's law. There are no point masses in nature. Thus, this formula is valid only to the extent in which the finite size of a mass can be ignored.* For our purposes, we will be satisfied with discussing only point masses. A second approximation has its origins in work by twentieth century physicists in which Newton's law is shown to be invalid as the velocities involved approach the speed of light. However, as long as the velocity of a mass is significantly less than the speed of light, Newton's law remains a good *approximation*. We emphasize the word approximation, for although mathematics is frequently treated as a science of exactness, mathematics is applied to models which only approximate the real world.

EXERCISES

2.1. Consider Fig. 2-2, which shows two masses (m_1 and m_2) attached to the opposite ends of a rigid (and massless) bar:

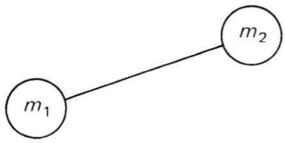

Figure 2-2.

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*Newton's second law can be applied to finite sized rigid bodies if $\vec{x}$, the position of the point mass, is replaced by $\vec{x}_{cm}$, the position of the center of mass (see exercise 2.1).

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