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ADVANCED LOGIC FOR APPLICATIONS

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PREFACE

This book is intended to be a survey of the most important results in mathematical logic for philosophers. It is a survey of results which have philosophical significance and it is intended to be accessible to philosophers. I have assumed the mathematical sophistication acquired in an introductory logic course or in reading a basic logic text. In addition to proving the most philosophically significant results in mathematical logic, I have attempted to illustrate various methods of proof. For example, the completeness of quantification theory is proved both constructively and non-constructively and relative advantages of each type of proof are discussed. Similarly, constructive and non-constructive versions of Gödel's first incompleteness theorem are given. I hope that the reader will develop facility with the methods of proof and also be caused by reflect on their differences.

I assume familiarity with quantification theory both in understanding the notations and in finding object language proofs. Strictly speaking the presentation is self-contained, but it would be very difficult for someone without background in the subject to follow the material from the beginning. This is necessary if the notes are to be accessible to readers who have had diverse backgrounds at a more elementary level. However, to make them accessible to readers with no background would require writing yet another introductory logic text. Numerous exercises have been included and many of these are integral parts of the proofs. This seems desirable since the purpose of the book is partly to provide the reader with the confidence and ability to go on to read more condensed material on his or her own. Some of the other examples are corollaries or interesting related theorems.

My intention is that the book should be useful both as a reference work and as a text for either self-teaching or classroom use. In order to preserve maximum flexibility, chapters have been kept independent of each other where possible. Figure a indicates graphically the

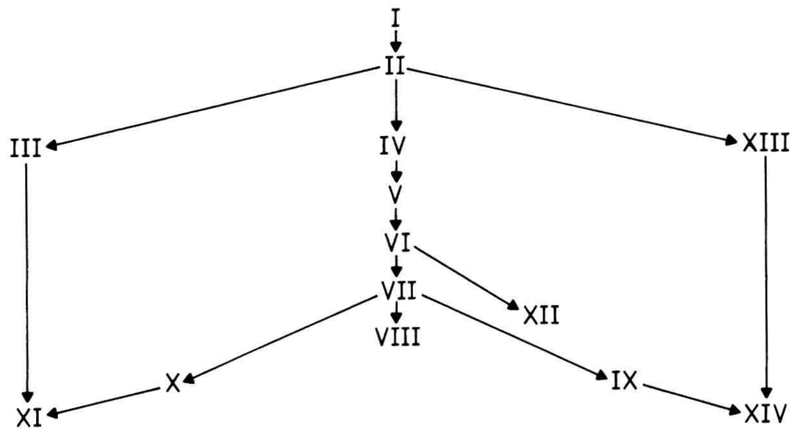

Figure a.

Material in any chapter presupposes material from those chapters which are connected to it by (a series of) arrows. For example, Chapter V presupposes I, II, and IV; Chapter XII presupposes I, II, IV, V, and VI; Chapter XIV presupposes I, II, IV, V, VI, VII, IX and also XIII.

dependencies between various chapters; I will try now to summarize the chapters and relations, and then to indicate the main types of course the book could be used for.

Chapter I isolates what I have called the Fundamental Theorem. In this theorem we characterize a particular type of set of formulas (called 'Henkin sets') and prove that these sets of formulas have interpretations. The definition of a Henkin set is entirely syntactic in the narrowest sense. That is, not only do we not mention anything about interpretations but we also make no reference to any axioms or rules. By relating the concept of a Henkin set to sets of formulas characterized in other ways, we derive the compactness and Skolem-Löwenheim theorems. Chapter I also includes a number of basic definitions required throughout the text.

Chapter II presents sets of axioms and rules of inference first for sentential calculus and then for full quantification theory and, using the Fundamental Theorem, proves the completeness of these systems. Chapter III presents an alternative formulation of first order quantification theory due to Gentzen; the completeness proof for this version of quantification theory is more closely connected with the particular rules of the system. As a consequence the proof is rather less general, but in compensation more useful corollaries concerning subsystems can be proved.

Chapters IV and V consider the extension of quantification theory to include identity and function symbols and prove some basic theorems about first order theories. The main theorems include the strong Löwenheim-Skolem theorem, the eliminability of function symbols and the partial eliminability of identity.

Chapter VI presents the general concepts and the main outline of the proof of the undecidability and incompleteness theorems. The purpose of this chapter is to give the reader overall grasp of the concepts and of the strategy of the proofs so that insight is not lost when all of the details are subsequently developed.

Chapter VII proves in detail the first Gödel theorem showing the incompleteness of any sufficiently rich number theory, Church's theorem concerning the undecidability of first order quantification theory and a number of other related theorems. Chapter VIII presents a detailed proof of Gödel's second incompleteness theorem establishing limitations on consistency proofs. Although the intuitive idea of this theorem can be stated as simply as that of the first theorem, a sufficiently accurate statement of the theorem is considerably more difficult. Considerable attention is paid to the conditions necessary for a statement to express the consistency of arithmetic. This chapter contains, to my knowledge, the first detailed textbook presentation of this theorem.

Chapter IX presents detailed proofs of Tarski's theorems, both negative and positive concerning the definability of truth. Although the topics here are somewhat independent of the previous chapters, the machinery used in proving the theorems depends heavily on previous chapters and hence cannot be read independently of them.

Chapter X extends the development of recursion theory which was begun in Chapter VII. The Kleene hierarchy is defined and various results are established concerning the undecidability of various sets and concerning the definability of recursive functions. Generalizations of Craig's theorem concerning types of axiomatizability and of Gödel's theorem are proved.

Chapter XI uses the recursive function theory just developed in order to provide a classical interpretation of intuitionistic logic and arithmetic (Kleene's recursive realizability interpretation). The independence of basic classical principles denied by intuitionists such as excluded middle and double negation is shown by means of this interpretation.

Chapter XII presents a system of second order logic, a generalization of first order logic in which quantification ranging over predicate positions is introduced. It is shown that Peano arithmetic theory based on this logic is categorical, unlike first order Peano arithmetic. It is also shown that the logic is not compact and has no recursive set of axioms. An alternative extension of first order logic which permits quantification over function symbols is also considered and shown equivalent to second order logic. Systems in which independent branches of first order quantifiers are permitted are also considered and their relation to first and second order theories is established.

Chapter XIII gives a detailed presentation of two alternative methods of formulating first order quantification in which the syntactic and semantic operations are more closely parallel. These systems are formulated in such a way that all assertions consist of equations between formulas which assert that the formulas are assigned the same truth conditions in the interpretation. In these systems the only rules of inference required are those for substitution of identities. The systems are shown to be equivalent to each other and to standard quantification theory in expressive power. In spite of their equivalence in expressive power, these systems embody a considerably different perspective on logic. From this perspective formulas are operations on sets of sequences and logic can be viewed as the study of these operations and their representation in various languages.

Chapter XIV considers a natural extension of the systems of the previous chapter which permits atomic predicates to be assigned sequences of varying length. It is shown that the standard quantification theory is properly contained in this system, that a particularly elegant definition of truth can be given in this theory, and that no recursive axiomatization of the logic exists.

Evidently this book could be used in various types of semester course emphasizing different aspects of non-elementary logic. A course in alternative forms of quantification theory could be given using Chapters I–V and Chapter XIII; a course on foundations of arithmetic could be given using Chapters I, II, IV, V–VII, X and XI; a course on first order theories could be given using Chapters I, II, IV–X; a course on alternatives to standard quantification theory could use Chapters I, II, IV–VII, X–XIV.

The Bibliographical Acknowledgements lists the original sources of the proofs and also contains indications of further material for the interested reader.

ACKNOWLEDGEMENTS

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CONTENTS

HENKIN SETS AND THE FUNDAMENTAL THEOREM

We will begin by proving a fundamental result which will be used repeatedly in the proofs of our major theorems. We will prove it for the full language of quantification theory even though some of our systems will have a restricted vocabulary. No change in the proof is required for the restricted vocabularies.

The full vocabulary of quantification theory consists of the logical particles $-$, $\supset$, $\wedge$, $\vee$, $\exists$, and $\forall$, the parentheses (,), an infinite list of individual variables $x_0, x_1, x_2, \ldots$, an infinite list of individual constants $c_0, c_1, c_2, \ldots$, and for each $n > 0$ an infinite list of n-place predicate letters $F^n_0, F^n_1, \ldots$.

A *term* is any individual variable or constant.

An *atomic formula* is an n-place predicate letter followed by a sequence of n terms.

A *sentential letter* is a 0-place predicate letter.

A sequence of symbols is a *formula* iff it is atomic or if it is of the form $(A \wedge B)$ or $-A$ or $(A \supset B)$ or $(A \vee B)$ or $(\forall v)A$ or $(\exists v)A$, where A and B are formulas and v is an individual variable.

This definition illustrates our practice of using A, B, C, D, E, A, $B_1, \ldots$ as metalinguistic variables for formulas v, $v_0 \ldots$ as metalinguistic variables for object language variables. In addition we use t, $t_1, \ldots$ as metalinguistic variables for terms.

The occurrences of a variable v in a formula $(\exists v)A$ or $(\forall v)A$ are *bound occurrences.* Occurrences of a variable which are not bound are *free.* We frequently abbreviate $(\forall v)A$ by $(v)A$.

We use the notation A^s_t to indicate the formula which results from substituting t for s in A provided that if s is a variable t is substituted only for free occurrences of s, and if t is a variable all new occurrences of t are free. If these conditions are not satisfied A^s_t is simply A.

EXERCISE 1. What formula is $((x_1)\ Fx_1c_1)^{c_1}_{x_1}$?

$$((x_1)Fx_1c_1)^{c_1}_{x_2}? \qquad ((x_1)Fx_1c_1)^{x_1}_{x_2}? \qquad ((x_2)Fx_1x_2)^{x_1}_{x_2}?$$

A model for the quantificational language is an ordered pair $\langle D, I\rangle$ where D is a non-empty set and I is a function such that

(1) for each constant c, $I(c) \in D$.

(2) for each predicate letter F^n_i, $I(F^n_i) \subseteq D^n$.

D^n is the set of n-tuples of objects in D. Note that D^0 has one element, the empty sequence $\langle\ \rangle$. Thus a 0-place predicate letter F^0_i can be assigned either Λ or $\{\langle\ \rangle\}$. The first corresponds to being assigned 'true' and the second 'false' in usual presentations. The present approach may look like a 'trick' but we will show in Chapter XIII why it is natural.

In order to define truth in a model we must first define satisfaction. Let α be a function which assigns an element of D to each individual variable and $I(c)$ to each constant. Such a function is said to be a *sequence in* $\langle D, I\rangle$ and we will use the metalinguistic variables α, β, γ, α_1, $\beta_1, \ldots$ to range over such sequences.

It will be useful to have a notation for the relation which holds between two sequences α and β when they agree on all variables except possibly v. We will write this as $\alpha \underset{v}{\approx} \beta$, and it means that for all $v' \neq v$, $\alpha(v') = \beta(v')$.

The relation α *satisfies* A *in* $\langle D, I\rangle$ is defined recursively:

α satisfies $F^n_{t_1 \ldots t_n}$ in $\langle D, I\rangle$ iff $\langle \alpha(t_1), \ldots \alpha(t_n)\rangle \in I(F^n)$

α satisfies $-A$ in $\langle D, I\rangle$ iff α does not satisfy A in $\langle D, I\rangle$

α satisfies $(A \wedge B)$ in $\langle D, I\rangle$ iff α satisfies A in $\langle D, I\rangle$ and satisfies B in $\langle D, I\rangle$

α satisfies $(A \vee B)$ in $\langle D, I\rangle$ iff α satisfies A in $\langle D, I\rangle$ or α satisfies B in $\langle D, I\rangle$

α satisfies $(A \supset B)$ in $\langle D, I\rangle$ iff α satisfies B in $\langle D, I\rangle$ or α does not satisfy A in $\langle D, I\rangle$

α satisfies $(v)A$ in $\langle D, I\rangle$ iff for all β, if $\alpha \underset{v}{\approx} \beta$ then β satisfies A in $\langle D, I\rangle$

> α satisfies $(\exists v)A$ in $\langle D, I\rangle$ iff for some β, $\alpha \underset{v}{\approx} \beta$ and β satisfies A in $\langle D, I\rangle$.

A formula A is *true in* $\langle D, I\rangle$ iff A is satisfied in $\langle D, I\rangle$ by all sequences in $\langle D, I\rangle$. A formula is *valid* iff it is true in all models. We often symbolize 'A is valid' as $\models A$. A sentence is *false* in a model iff its negation is true in that model. Note that there are formulas and models such that the formula is neither true nor false in the model. We define a formula to be a *closed formula* or a *sentence* iff it has no free variables. Closed formulas are true or false in each model.

EXERCISE 2. Give an example of a formula A and model $\langle D, I\rangle$ such that A is neither true nor false in $\langle D, I\rangle$.

EXERCISE 3. Show that if A is a closed formula and $\langle D, I\rangle$ is a model then A is either true or false in $\langle D, I\rangle$.

A formula is *satisfiable* iff it is satisfied by some α in some model. A set of formulas is *simultaneously satisfiable* iff there is an α and a model such that α satisfies all of those formulas in that model. We will use Γ with and without subscripts as a metalanguage variable ranging over sets of formulas. We will use $\in$ in its usual set theoretic sense of membership.

A formula A is a semantic consequence of a set of formulas Γ iff every sequence and model that simultaneously satisfy Γ also satisfy A. This is equivalent to saying that $\Gamma \cup (-A)$ is not simultaneously satisfiable. We will often simply speak of 'consequence' meaning 'semantic consequence', and we will symbolize it as $\Gamma \models A$.

Some other conventions will be useful. We will speak of A having a model M or of M being a model for A; this means that A is satisfiable in M. Similarly we will speak of sets of formulas having a model meaning that they are simultaneously satisfied in some model. Finally we will often abbreviate simultaneously satisfiable as s.s.

Our first theorem is an intuitively plausible one which is needed frequently in our proofs. It tells us that if two formulas are alike except for their constants and free variables then if α and β assign the same things to corresponding terms then α satisfies the one formula iff β satisfies the other. More rigorously,

THEOREM. *For any model and any sequences α, β, and any formulas A, B, if*

(i) $t_1 \ldots t_n$ *do not occur in* B

(ii) $t_{n+1} \ldots t_{2n}$ *are variables which do not appear in* A

(iii) B *is* $A^{t_1 \ldots t_n}_{t_{n+1} \ldots t_{2n}}$

(iv) $\alpha(v) = \beta(v)$ *unless* $v = t_1$ *or* $\ldots$ *or* $v = t_{2n}$

(v) $\alpha(t_i) = \beta(t_{n+i})$,

then α satisfies A iff β satisfies B.

Proof. By induction on the order of formulas. An atomic formula is of order 1. If A is of order n then $-A$, $(v)A$ and $\exists vA$ are of order $n+1$. If n is the maximum of the orders of A and B then the order of $(A \wedge B)$, and $(A \supset B)$ is $n+1$.

If A is of order 1 then A is atomic and, by construction, α and β assign the same elements to the corresponding terms of A and B.

We now assume the theorem holds for orders $<n$ and show that it holds for n as well. If A is a negation, disjunction, conjunction or implication the fact to be shown follows immediately from the induction hypothesis and the definition of satisfaction. For example, if A is $-C$ then B is $-D$ where C and D satisfy the conditions of the theorem and are of order $n-1$. Therefore, α satisfies C iff β satisfies D and so α satisfies $-C$ iff β satisfies $-D$.

If A is $(v)C$ then B is $(v)D$ where C and D meet the conditions of the theorem. If $\alpha' \underset{v}{\approx} \alpha$ and $\beta' \underset{v}{\approx} \beta$ and $\alpha'(v) = \beta'(v)$ then by induction hypothesis α' sat C iff β' sat D. But then α sat $(v)C$ iff for all $\alpha' \underset{v}{\approx} \alpha$, α' satisfies C iff for all $\beta' \underset{v}{\approx} \beta$, β' satisfies D iff β satisfies $(v)D$. A similar argument establishes that α satisfies $\exists vC$ iff β satisfies $\exists vD$, if A is $\exists vC$ and B is $\exists vD$.

Intuitively, the formulas $(x_1)Fx_1x_3$ and $(x_{99})Fx_{99}x_3$ express the same thing. In general, we will say that A and B are alphabetic variants of one another if they are exactly alike except that occurrences of one or more bound variables in A are replaced by corresponding occurrences of bound variables in B. By corresponding occurrences we mean to require that all free variables of A are also free in B and that distinct bound variables of A are replaced by distinct bound variables in B.

EXERCISE 4. Which of the following are alphabetic variants of

$$(x_1)(x_3)Fx_1x_2x_3? \qquad (x_1)(x_4)Fx_1x_2x_4 \qquad (x_4)(x_3)Fx_4x_2x_3$$

$$(x_1)(x_3)Fx_1x_4x_3 \qquad (x_1)(x_1)Fx_1x_2x_1 \qquad (x_1)(x_2)Fx_1x_2x_2.$$

EXERCISE 5. Show that if A and B are alphabetic variants of one another, a sequence satisfies A iff it satisfies B.

We often want to show that a formula or set of formulas has a model. We now prove an important theorem which will be a basic tool throughout the book and we will give two applications. We are going to define a particular type of set of formulas, *Henkin sets*, and we will prove that every Henkin set has a model. Then to show that a given set of formulas has a model, it will only be necessary to show that the set in question is a subset of some Henkin set. The definition of a Henkin set is closely modeled on the definition of satisfaction in a model.

Γ is a Henkin set iff

(a) for all A either $A \in \Gamma$ or $-A \in \Gamma$
(b) for no A, $A \in \Gamma$ and $-A \in \Gamma$
(c) for all B and A, $(A \wedge B) \in \Gamma$ iff $A \in \Gamma$ and $B \in \Gamma$
(d) for all B and A, $(A \vee B) \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$
(e) for all B and A, $(A \supset B) \in \Gamma$ iff $A \neq \Gamma$ or $B \in \Gamma$
(f) If $A \in \Gamma$, then all formulas which are alphabetic variants of A are in Γ
(g) for all A, v, $(v)A \in \Gamma$ iff, for all terms t, $A_t^v \in \Gamma$
(h) for all A and v, $(\exists v)A \in \Gamma$ iff for some term t, $A_t^v \in \Gamma$

EXERCISE 6. Let $\langle D, I\rangle$ be a model in which every element of the domain is assigned to some constant. Show that for any α, $\{A$: α satisfies A in $\langle D, I\rangle\}$ is a Henkin set.

EXERCISE 7. Let Γ be a set of formulas such that