New Advances in Transcendence Theory

EDITED BY

Alan Baker

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PREFACE

A highly successful Symposium on Transcendental Number Theory was held under the auspices of the London Mathematical Society at the University of Durham in July, 1986, and the present volume is an account of the proceedings of that meeting. Some fifty mathematicians were present, including most of the leading specialists in the field, and the lectures reflected the remarkable research activity that has taken place in this area in recent years. Indeed, as became apparent, the evolution of transcendence, since the 1960s, into a fertile theory with numerous and widespread applications has been one of the most exciting and important developments of modern mathematics. The conference programme, though comprehensive, was intended to be in no way overcrowded, and it was particularly aimed to create a relaxed atmosphere for the free exchange of ideas. This seems to have worked out well; in fact much valuable material was presented for future study and some original theorems were obtained through informal collaboration during the meeting itself. The invited participants from the USSR were alas unable to come to Durham but they communicated reports subsequently and the editor is grateful to them and indeed to all the distinguished authors for contributing so admirably to this volume.

A conference with a similar theme was held in Cambridge some ten years ago and the proceedings were published under the title *Transcendence Theory: Advances and Applications* (Academic Press, 1977); the present work forms a natural sequel. Again, many papers are concerned with the theory of linear forms in the logarithms of algebraic numbers. In particular, the memoirs of Wüstholz and of Philippon and Waldschmidt both contain definitive results in this context; they eliminate a second order factor from the inequalities that I established at the time of the meeting in Cambridge, and the arguments rest ultimately on the spectacular progress that has been made in recent years, most notably by Wüstholz, concerning multiplicity estimates on group varieties. Studies in the area were initiated by Nesterenko and some new related estimates are given in his paper here. The articles of Bertrand and of Masser illuminate other aspects of proofs in this field, highlighting, for instance, the extensive connections with Kummer theory and elliptic curves. Links with the classic works of Gelfond and Schneider are described in the papers of Feldman and of Waldschmidt, and the current status of the p-adic theory is discussed by Kunrui Yu.

Another major topic is the application of transcendence theory to the study of Diophantine equations. Here the very substantial paper on S-unit equations by Evertse, Győry, Stewart and Tijdeman, and the associated work on decomposable form equations by Evertse and Győry, are particularly welcome. There are, moreover, valuable articles on exponential Diophantine equations by Shorey, on the Thue equation by Schmidt, and on equations over function fields by Mason and by Brindza. The article by Baker and Stewart also relates to Diophantine equations; it is shown that the theory of linear forms in logarithms can be greatly streamlined in certain instances so as to yield surprisingly good numerical bounds.

A subject that has plainly attracted a great deal of research in recent years is the transcendence theory of classical functions, with particular interest focused on hypergeometric functions, on E-functions and on G-functions. The excellent papers by Beukers, Beukers and Wolfart, Galochkin, Shidlovsky and Sprindžuk all cover aspects of this topic. They refer to many recent results, and, taken together, they provide the most complete survey of the field available to date.

Furthermore, this by no means exhausts the range of material that can be found here. Indeed, the paper of Bernik is concerned with the metrical theory of transcendence, an area to which he has made some striking advances; the paper of Brownawell is concerned with the remarkable relation between Hilbert's irreducibility theorem and transcendence; the paper of Erdös is concerned with the questions of irrationality and transcendence; the paper of Loxton is concerned with automata and transcendence and, in particular, with new problems connected with the celebrated Mahler method; the paper of Odoni is concerned with modular forms and transcendence and furnishes the answer to a question of Serre; and the paper of Schinzel continues his fine series of studies on reducibility of polynomials and shows that there is a useful role for transcendence here too. It seems probable that the work as a whole will be of considerable influence in determining the future direction of the theory.

The Symposium was funded by a grant from the Science and Engineering Research Council and this support is acknowledged with gratitude. My colleague, Dr R. C. Mason handled all the domestic and financial arrangements and there is no doubt that the success of the meeting was due in no small measure to his excellent work. The co-operation of Prof. P. Higgins and indeed of all our mathematical colleagues in Durham was invaluable, and particular thanks should be expressed to the Bursar and his staff at Grey College for their helpfulness throughout. Thanks are also due to Dr A. Harris for generously taking on the task of translating articles from Russian into English, to Prof. J. W. S. Cassels for additional advice in this respect, and to Dr D. Tranah of Cambridge University Press for his kind and patient assistance at all stages of production of this volume.

Cambridge, 1987

A. B.

Added in proof. It is with much sadness that the editor records here the death of Prof. V. G. Sprindžuk in July, 1987. His passing is a great loss to mathematics, and we shall remember especially his important contributions to Transcendence Theory.

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CONTENTS

1

ON EFFECTIVE APPROXIMATIONS TO CUBIC IRRATIONALS

A. Baker and C. L. Stewart*

1. Introduction

The problem of obtaining effective measures of irrationality for algebraic irrationals has recently attracted considerable attention. The first result in this field was discovered by Baker [1], [2] in 1964. He used properties of hypergeometric series to obtain effective results for certain fractional powers of rationals. It was shown, in particular, that for all rationals p/q with $q > 0$ we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^\kappa}, \tag{1}$$

where $\alpha = \sqrt[3]{2}$, $c = 10^{-6}$ and $\kappa = 2.955$. A similar result was established for instance for $\alpha = \sqrt[3]{19}$ with $c = 10^{-9}$ and $\kappa = 2.56$. This work was recently refined by Chudnovsky [11]; by a careful study of the Padé approximants occurring in the hypergeometric method he obtained more precise values for κ and consequently he was able to deal with a wider range of algebraic numbers. Chudnovsky left the values for c occurring in his results unspecified but these have recently been established in some special cases by Easton [13]. Easton has shown in particular that (1) holds with $\alpha = \sqrt[3]{28}$, $c = 7.5 \times 10^{-7}$ and $\kappa = 2.9$.

The results above improved upon the relatively crude inequality of Liouville established in 1844 to the effect that (1) holds for any algebraic number α, where $\kappa = n$, $n \geq 1$, the degree of α and c is an effectively computable positive number depending only on α. The first general effective improvement on Liouville's theorem was obtained by Baker [3] in 1968 using the theory of linear forms in the logarithms of algebraic

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numbers. A more precise version of the result was obtained subsequently by Feldman [14] and an explicit formulation of the theorem has recently been given by Győry and Papp [15]. In the present paper we shall sharpen the result of Győry and Papp in the case of cube roots of integers. We shall prove the following result.

Theorem 1. *Let a be a positive integer not a perfect cube, and let $\alpha = \sqrt[3]{a}$. Further let ϵ be the fundamental unit in the field $\mathbf{Q}(\sqrt[3]{a})$. Then (1) holds for all rational numbers p/q, $q > 0$, with $c = 1/(3ac_1)$ and $\kappa = 3 - 1/c_2$, where*

$$c_1 = \epsilon^{(50 \log\log \epsilon)^2}, \qquad c_2 = 10^{12} \log \epsilon. \tag{2}$$

Here $\mathbf{Q}$ denotes, as usual, the field of rational numbers and by the fundamental unit ϵ in $\mathbf{Q}(\sqrt[3]{a})$ we mean the smallest unit in the field larger than 1. Note that some authors adopt the alternative convention that the fundamental unit lies between 0 and 1. The result of Győry and Papp mentioned above yields a theorem similar to Theorem 1 but with

$$c_2 = 300^{60} \log \epsilon (\log\log \epsilon)^2 \tag{3}$$

and with a value for c_1 slightly greater than $(40a)^6\epsilon$. In both (2) and (3) we have made use of the fact, established in §2 below, that $\log \epsilon > 1$ for all fields $\mathbf{Q}(\sqrt[3]{a})$. Although our value for c_2 improves substantially on (3), the value for κ that it furnishes is far from the exponent $2+\delta$, $\delta > 0$, occurring in the Thue-Siegel-Roth theorem. As is well known the latter theorem is ineffective, that is, it does not provide an explicit value for the constant c in (1). But Bombieri [8] and Bombieri and Mueller [9] have recently shown that in certain special cases effective results can in fact be derived from the Thue-Siegel method. Nevertheless the restrictions attaching to α in their work are very stringent at present.

The inequality established in Theorem 1 is essentially equivalent to an upper bound for the solutions of the Diophantine equation

$$x^3 - ay^3 = n. \tag{4}$$

We have the following result.

Theorem 2. *Let a and n be positive integers with a not a perfect cube. Then all solutions in integers x and y of (4) satisfy*

$$\max(|x|, |y|) < (c_1 n)^{c_2},$$

where c_1 and c_2 are given by (2).

In order to derive Theorem 1 from Theorem 2 we denote by p/q, $q > 0$, any rational number and we suppose that $|\alpha - p/q| \leq c$; then $|p/q| \leq \alpha + c$, whence

$$\left|\alpha^2 + \alpha(p/q) + (p/q)^2\right| \leq 3\alpha^2 + 3\alpha c + c^2 \leq 3a.$$

This gives

$$\left|a - (p/q)^3\right| \leq 3a\left|\alpha - p/q\right|. \tag{5}$$

We now apply Theorem 2 with $n = |p^3 - aq^3|$ and conclude that $q < (c_1 n)^{c_2}$ whence $n > (1/c_1)q^{1/c_2}$. By (5) we have $\left|\alpha - p/q\right| \geq n/(3aq^3)$ and our result follows.

The proof of Theorem 2 is based essentially on the methods of [3] and [4]. In particular we reduce the problem to the study of a linear form in three logarithms and we ultimately establish the bound $2 \cdot 10^{12} \log(c_1 n)$ for the size of the integer coefficients in that form. Our exposition will follow the general pattern of the earlier papers but we shall use a simplified auxiliary function, and also a more efficient extrapolation procedure to which Kummer theory can be applied directly. The work here together with the technique of Baker and Davenport [6] would enable the complete list of solutions of (4) to be computed for any moderately sized a and n. Indeed we have $\log \epsilon < (0.37)d^{1/2}(\log d)^2$ where d is the absolute value of the discriminant of $\mathbf{Q}(\sqrt[3]{a})$ (see [18]); thus, since $d \leq 27a^2$ we obtain, for $a > 3$,

$$\log c_1 \leq (50 \log d)^2 \log \epsilon \leq (37 \log a)^4 a.$$

Hence if, for example, $a \leq 10^3$ and $\log n \leq 10^{10}$ then the coefficients of the logarithms in the linear form will have sizes at most 10^{25}.

As a particular instance of Theorem 1 we take $\alpha = \sqrt[3]{5}$; this is the smallest cube root not covered by the papers employing the hypergeometric method. Then $\epsilon = 41 + 24\alpha + 14\alpha^2$ (see [10], Table 2, p. 270) and $\log \epsilon < 5$. Hence we conclude that (1) holds with $c = 10^{-12900}$ and

$$\kappa = 2.9999999999998.$$

We should like to express our thanks to Professor D. Djokovic for his generous assistance in the computational work referred to in §3. The latter was carried out while the first author was visiting the University of Waterloo and he is grateful for their hospitality.

2. Preliminary lemmas

We shall require modified forms of two classical lemmas in transcendence theory. First we obtain the following sharpening of Lemma 4 of Baker and Stark [7].

Lemma 1. *Suppose that α, β are elements of an algebraic number field and that for some positive integer p we have $\alpha = \beta^p$. If a, b are the leading coefficients in the field polynomials defining α, β respectively then $b \le a^{1/p}$.*

Here the field polynomials are, as usual, powers of the minimal polynomials with degree D, where D denotes the degree of the field. Lemma 4 of [7] gives the weaker inequality $b \le a^{D/p}$, where a denotes any non-zero integer such that $a\alpha$ is an algebraic integer.

Proof. Let $\alpha^{(1)}, \dots, \alpha^{(D)}$ and $\beta^{(1)}, \dots, \beta^{(D)}$ be the field conjugates of α and β respectively. Then b is the least positive integer such that

$$f(x) = b(x - \beta^{(1)}) \dots (x - \beta^{(D)})$$

has rational integer coefficients. We write

$$g(x) = a(x^p - \alpha^{(1)}) \dots (x^p - \alpha^{(D)}), \qquad h(x) = \prod_{j=1}^{p} f(xe^{2\pi ij/p}).$$

Since, by hypothesis, $\alpha = \beta^p$ we have

$$b^p g(x) = (-1)^{D(p+1)} a h(x).$$

Arguing as in [7] we deduce from the algebraic generalization of Gauss' lemma that $h(x)$ has relatively prime rational integer coefficients. But $g(x)$ also has rational integer coefficients and so b^p divides a, whence $b \le a^{1/p}$ as required.

Secondly, we shall establish a version of Siegel's lemma appropriate to our work here. We shall adapt the result of Dobrowolski [12] so as to deal with linear forms with arbitrary algebraic coefficients, not merely algebraic integers. Obviously it would suffice to multiply through each equation by a suitable common denominator but this would be too crude for our purpose. In order to state the lemma, we define K to be an algebraic number field with degree n over $\mathbb{Q}$ and we let $\sigma_1, \dots, \sigma_n$ be the embeddings of K in the complex numbers. Further we signify by b_{ij}, $1 \le i \le N$, $1 \le j \le M$, elements of K such that for each j not all b_{ij},

$1 \le i \le N$, are zero. We now define c_j, $1 \le j \le M$, to be a positive integer such that

$$c_j \sigma_1(b_{i_1,j}) \ldots \sigma_n(b_{i_n,j})$$

is an algebraic integer for all choices of $i_1, \ldots, i_n$.

Lemma 2. *If $N > nM$ then the system of equations*

$$\sum_{i=1}^{N} b_{ij} x_i = 0, \qquad 1 \le j \le M,$$

has a solution in rational integers $x_1, \ldots, x_N$, not all 0, with absolute values at most

$$Y = \left(2\sqrt{2}(N+1) Z^{1/(nM)}\right)^{nM/(N-nM)},$$

where

$$Z = \prod_{j=1}^{M} \Bigl(c_j \prod_{k=1}^{n} \max_i \bigl| \sigma_k(b_{ij}) \bigr| \Bigr).$$

Proof. The proof follows almost verbatim that of Dobrowolski [12]. the main idea is to select rational integers $x_1, \ldots, x_N$ by the box principle such that

$$\Bigl| c_j N_{K/\mathbf{Q}} \Bigl(\sum_i b_{ij} x_i \Bigr) \Bigr| < 1, \qquad 1 \le j \le M.$$

This differs from [12] by virtue of the presence of c_j; our definition of c_j ensures that the expression on the left of the above inequality is a rational integer. The only significant modification in the proof concerns the quantity

$$C_j = \Bigl(c_j \prod_{k=1}^{n} \max_i \bigl| \sigma_k(b_{ij}) \bigr| \Bigr)^{1/n}$$

which now includes c_j. This leads to the definition

$$\ell_j = (Y^N / Z)^{1/(nM)} C_j,$$

which gives

$$2\sqrt{2}(N+1) Y C_j - \ell_j = 0$$

as in [12]. Further, as there, we note that $C_j \geq 1$ and hence also $Y \geq 1$; this follows from our definition of c_j and the assumption that, for each j, not all b_{ij} are zero.†

We now record three lemmas that will be needed later. Lemma 3 is classical Kummer theory; for a proof see Baker and Stark [7]. Lemma 4 is a famous result of Delaunay and Nagell; for a proof see Nagell [17]. Lemma 5 is due to Ljunggren [16].

Lemma 3. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of an algebraic number field K and let $\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p}$ denote fixed pth roots for some prime p. Further, let $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of K' of degree p or we have*

$$\alpha_n = \alpha_1^{j_1} \ldots \alpha_{n-1}^{j_{n-1}} \gamma^p$$

for some γ in K and some integers $j_1, \ldots, j_{n-1}$ with $0 \leq j_\ell < p$.

Lemma 4. *Let a be a positive integer, not a perfect cube. The equation*

$$x^3 - ay^3 = 1$$

has at most one solution in integers x, y with $y \neq 0$ and, for this, $x - y\sqrt[3]{a}$ is given by either $1/\epsilon$ or $1/\epsilon^2$, where ϵ is the fundamental unit of $\mathbf{Q}(\sqrt[3]{a})$ as in §1.

Lemma 5. *Let A, B, C be positive integers with $C = 1$ or $C = 3$ and suppose that A and B are > 1 when $C = 1$. Suppose further that AB is not divisible by 3 when $C = 3$. Then the equation*

$$Ax^3 + By^3 = C$$

has at most one solution in integers x, y and for this, $C^{-1}\left(x\sqrt[3]{A} + y\sqrt[3]{B}\right)^3$ is either $1/\eta$ or $1/\eta^2$ where η is the fundamental unit in $\mathbf{Q}\left(\sqrt[3]{(AB^2)}\right)$. The only exception is the equation $2x^3 + y^3 = 3$ which has two solutions, namely $x = y = 1$ and $x = 4$, $y = -5$.

Note that if the condition in Lemma 5 that AB be not divisible by 3 when $C = 3$ is violated then the equation reduces to an equation with

† Professor Vaaler has pointed out to us that the result can also be obtained from Theorem 9 of Bombieri and Vaaler, "On Siegel's Lemma", *Invent. Math.* **73** (1983), 11–32, and in fact with $\sqrt{N}$ in place of $2\sqrt{2}(N+1)$.