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Preface

The present book is intended to be a systematic text on topological vector spaces and presupposes familiarity with the elements of general topology and linear algebra. The author has found it unnecessary to rederive these results, since they are equally basic for many other areas of mathematics, and every beginning graduate student is likely to have made their acquaintance. Similarly, the elementary facts on Hilbert and Banach spaces are widely known and are not discussed in detail in this book, which is mainly addressed to those readers who have attained and wish to get beyond the introductory level.

The book has its origin in courses given by the author at Washington State University, the University of Michigan, and the University of Tübingen in the years 1958–1963. At that time there existed no reasonably complete text on topological vector spaces in English, and there seemed to be a genuine need for a book on this subject. This situation changed in 1963 with the appearance of the book by Kelley, Namioka *et al.* [1] which, through its many elegant proofs, has had some influence on the final draft of this manuscript. Yet the two books appear to be sufficiently different in spirit and subject matter to justify the publication of this manuscript; in particular, the present book includes a discussion of topological tensor products, nuclear spaces, ordered topological vector spaces, and an appendix on positive operators. The author is also glad to acknowledge the strong influence of Bourbaki, whose monograph [7], [8] was (before the publication of Köthe [5]) the only modern treatment of topological vector spaces in printed form.

A few words should be said about the organization of the book. There is a preliminary chapter called "Prerequisites," which is a survey aimed at clarifying the terminology to be used and at recalling basic definitions and facts to the reader's mind. Each of the five following chapters, as well as the Appendix, is divided into sections. In each section, propositions are marked u.v, where u is the section number, v the proposition number within the section. Propositions of special importance are additionally marked "Theorem." Cross references within the chapter are (u.v), outside the chapter (r, u.v), where r (roman numeral) is the number of the chapter referred to. Each chapter is preceded by an introduction and followed by exercises. These "Exercises" (a total of 142) are devoted to further results and supplements, in particular, to examples and counter-examples. They are not meant to be worked out one after the other, but every reader should take notice of them because of their informative value. We have refrained from marking some of them as difficult, because the difficulty of a given problem is a highly subjective matter. However, hints have been given where it seemed appropriate, and occasional references indicate literature that may be needed, or at least helpful. The bibliography, far from being complete, contains (with few exceptions) only those items that are referred to in the text.

I wish to thank A. Pietsch for reading the entire manuscript, and A. L. Peressini and B. J. Walsh for reading parts of it. My special thanks are extended to H. Lotz for a close examination of the entire manuscript, and for many valuable discussions. Finally, I am indebted to H. Lotz and A. L. Peressini for reading the proofs, and to the publisher for their care and cooperation.

H. H. S.

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PREREQUISITES

A formal prerequisite for an intelligent reading of this book is familiarity with the most basic facts of set theory, general topology, and linear algebra. The purpose of this preliminary section is not to establish these results but to clarify terminology and notation, and to give the reader a survey of the material that will be assumed as known in the sequel. In addition, some of the literature is pointed out where adequate information and further references can be found.

Throughout the book, statements intended to represent definitions are distinguished by setting the term being defined in bold face characters.

A. SETS AND ORDER

1. *Sets and Subsets.* Let X, Y be sets. We use the standard notations $x \in X$ for "x is an element of X", $X \subset Y$ (or $Y \supset X$) for "X is a subset of Y", $X = Y$ for "$X \subset Y$ and $Y \supset X$". If (p) is a proposition in terms of given relations on X, the subset of all $x \in X$ for which (p) is true is denoted by $\{x \in X: (p)x\}$ or, if no confusion is likely to occur, by $\{x: (p)x\}$. $x \notin X$ means "x is not an element of X". The **complement** of X relative to Y is the set $\{x \in Y: x \notin X\}$, and denoted by $Y \sim X$. The empty set is denoted by $\emptyset$ and considered to be a finite set; the set (**singleton**) containing the single element x is denoted by $\{x\}$. If (p_1), (p_2) are propositions in terms of given relations on X, $(p_1) \Rightarrow (p_2)$ means "(p_1) implies (p_2)", and $(p_1) \Leftrightarrow (p_2)$ means "(p_1) is equivalent with (p_2)". The set of all subsets of X is denoted by $\mathfrak{P}(X)$.

2. *Mappings.* A mapping f of X into Y is denoted by $f: X \to Y$ or by $x \to f(x)$. X is called the **domain** of f, the image of X under f, the **range** of f; the **graph** of f is the subset $G_f = \{(x, f(x)): x \in X\}$ of $X \times Y$. The mapping of the set $\mathfrak{P}(X)$ of all subsets of X into $\mathfrak{P}(Y)$ that is associated with f, is also denoted by f; that is, for any $A \subset X$ we write $f(A)$ to denote the set

$\{f(x) : x \in A\} \subset Y$. The associated map of $\mathfrak{P}(Y)$ into $\mathfrak{P}(X)$ is denoted by f^{-1}; thus for any $B \subset Y$, $f^{-1}(B) = \{x \in X: f(x) \in B\}$. If $B = \{b\}$, we write $f^{-1}(b)$ in place of the clumsier (but more precise) notation $f^{-1}(\{b\})$. If $f: X \to Y$ and $g: Y \to Z$ are maps, the composition map $x \to g(f(x))$ is denoted by $g \circ f$.

A map $f: X \to Y$ is **biunivocal (one-to-one, injective)** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$; it is **onto** Y **(surjective)** if $f(X) = Y$. A map f which is both injective and surjective is called **bijective** (or a **bijection**).

If $f: X \to Y$ is a map and $A \subset X$, the map $g : A \to Y$ defined by $g(x) = f(x)$ whenever $x \in A$ is called the **restriction** of f to A and frequently denoted by f_A. Conversely, f is called an **extension** of g (to X with values in Y).

3. *Families.* If A is a *non-empty* set and X is a set, a mapping $\alpha \to x(\alpha)$ of A into X is also called a **family** in X; in practice, the term family is used for mappings whose domain A enters only in terms of its set theoretic properties (i.e., cardinality and possibly order). One writes, in this case, x_α for $x(\alpha)$ and denotes the family by $\{x_\alpha: \alpha \in \mathrm{A}\}$. Thus every non-empty set X can be viewed as the family (identity map) $x \to x (x \in X)$; but it is important to notice that if $\{x_\alpha: \alpha \in \mathrm{A}\}$ is a family in X, then $\alpha \neq \beta$ does not imply $x_\alpha \neq x_\beta$. A **sequence** is a family $\{x_n: n \in N\}$, $N = \{1, 2, 3, \ldots\}$ denoting the set of natural numbers. If confusion with singletons is unlikely and the domain (index set) A is clear from the context, a family will sometimes be denoted by $\{x_\alpha\}$ (in particular, a sequence by $\{x_n\}$).

4. *Set Operations.* Let $\{X_\alpha: \alpha \in \mathrm{A}\}$ be a family of sets. For the union of this family, we use the notations $\bigcup\{X_\alpha: \alpha \in \mathrm{A}\}$, $\bigcup_{\alpha \in \mathrm{A}} X_\alpha$, or briefly $\bigcup_\alpha X_\alpha$ if the index set A is clear from the context. If $\{X_n: n \in N\}$ is a sequence of sets we also write $\bigcup_1^\infty X_n$, and if $\{X_1, \ldots, X_k\}$ is a finite family of sets we write $\bigcup_1^k X_n$ or $X_1 \cup X_2 \cup \ldots \cup X_k$. Similar notations are used for intersections and Cartesian products, with $\bigcup$ replaced by $\bigcap$ and $\prod$ respectively. If $\{X_\alpha: \alpha \in \mathrm{A}\}$ is a family such that $X_\alpha = X$ for all $\alpha \in \mathrm{A}$, the product $\prod_\alpha X_\alpha$ is also denoted by X^{A}.

If R is an equivalence relation (i.e., a reflexive, symmetric transitive binary relation) on the set X, the set of equivalence classes (the **quotient set**) by R is denoted by X/R. The map $x \to \hat{x}$ (also denoted by $x \to [x]$) which orders to each x its equivalence class $\hat{x}$ (or $[x]$), is called the **canonical** (or **quotient**) map of X onto X/R.

5. *Orderings.* An **ordering (order structure, order)** on a set X is a binary relation R, usually denoted by $\leqq$, on X which is reflexive, transitive, and anti-symmetric ($x \leqq y$ and $y \leqq x$ imply $x = y$). The set X endowed with an order $\leqq$ is called an **ordered** set. We write $y \geqq x$ to mean $x \leqq y$, and $x < y$ to mean $x \leqq y$ but $x \neq y$ (similarly for $x > y$). If R_1 and R_2 are orderings of X, we say that R_1 is **finer** than R_2 (or that R_2 is **coarser** than R_1) if $x(R_1)y$ implies $x(R_2)y$. (Note that this defines an ordering on the set of all orderings of X.)

Let $(X, \leqq)$ be an ordered set. A subset A of X is **majorized** if there exists $a_0 \in X$ such that $a \leqq a_0$ whenever $a \in A$; a_0 is a **majorant (upper bound)** of A. Dually, A is **minorized** by a_0 if $a_0 \leqq a$ whenever $a \in A$; then a_0 is a **minorant (lower bound)** of A. A subset A which is both majorized and minorized, is called **order bounded**. If A is majorized and there exists a majorant a_0 such that $a_0 \leqq b$ for any majorant b of A, then a_0 is unique and called the **supremum (least upper bound)** of A; the notation is $a_0 = \sup A$. In a dual fashion, one defines the **infimum (greatest lower bound)** of A, to be denoted by $\inf A$. For each pair $(x, y) \in X \times X$, the supremum and infimum of the set $\{x, y\}$ (whenever they exist) are denoted by $\sup(x, y)$ and $\inf(x, y)$ respectively. $(X, \leqq)$ is called a **lattice** if for each pair (x, y), $\sup(x, y)$ and $\inf(x, y)$ exist, and $(X, \leqq)$ is called a **complete lattice** if $\sup A$ and $\inf A$ exist for every non-empty subset $A \subset X$. (In general we avoid this latter terminology because of the possible confusion with uniform completeness.) $(X, \leqq)$ is **totally ordered** if for each pair (x, y), at least one of the relations $x \leqq y$ and $y \leqq x$ is true. An element $x \in X$ is **maximal** if $x \leqq y$ implies $x = y$.

Let $(X, \leqq)$ be a *non-empty* ordered set. X is called **directed** under $\leqq$ (briefly, directed $(\leqq)$) if every subset $\{x, y\}$ (hence each finite subset) possesses an upper bound. If $x_0 \in X$, the subset $\{x \in X : x_0 \leqq x\}$ is called a **section** of X (more precisely, the section of X **generated** by x_0). A family $\{y_\alpha : \alpha \in \mathrm{A}\}$ is **directed** if A is a directed set; the **sections** of a directed family are the subfamilies $\{y_\alpha : \alpha_0 \leqq \alpha\}$, for any $\alpha_0 \in \mathrm{A}$.

Finally, an ordered set X is **inductively ordered** if each totally ordered subset possesses an upper bound. In each inductively ordered set, there exist maximal elements (Zorn's lemma). In most applications of Zorn's lemma, the set in question is a family of subsets of a set S, ordered by set theoretical inclusion $\subset$.

6. *Filters*. Let X be a set. A set $\mathfrak{F}$ of subsets of X is called a **filter** on X if it satisfies the following axioms:

(1) *$\mathfrak{F} \neq \emptyset$ and $\emptyset \notin \mathfrak{F}$.*
(2) *$F \in \mathfrak{F}$ and $F \subset G \subset X$ implies $G \in \mathfrak{F}$.*
(3) *$F \in \mathfrak{F}$ and $G \in \mathfrak{F}$ implies $F \cap G \in \mathfrak{F}$.*

A set $\mathfrak{B}$ of subsets of X is a **filter base** if (1′) $\mathfrak{B} \neq \emptyset$ and $\emptyset \notin \mathfrak{B}$, and (2′) if $B_1 \in \mathfrak{B}$ and $B_2 \in \mathfrak{B}$ there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subset B_1 \cap B_2$. Every filter base $\mathfrak{B}$ generates a unique filter $\mathfrak{F}$ on X such that $F \in \mathfrak{F}$ if and only if $B \subset F$ for at least one $B \in \mathfrak{B}$; $\mathfrak{B}$ is called a base of the filter $\mathfrak{F}$. The set of all filters on a non-empty set X is inductively ordered by the relation $\mathfrak{F}_1 \subset \mathfrak{F}_2$ (set theoretic inclusion of $\mathfrak{P}(X)$); $\mathfrak{F}_1 \subset \mathfrak{F}_2$ is expressed by saying that $\mathfrak{F}_1$ is **coarser** than $\mathfrak{F}_2$, or that $\mathfrak{F}_2$ is **finer** than $\mathfrak{F}_1$. Every filter on X which is maximal with respect to this ordering, is called an **ultrafilter** on X; by Zorn's lemma, for each filter $\mathfrak{F}$ on X there exists an ultrafilter finer than $\mathfrak{F}$. If $\{x_\alpha : \alpha \in \mathrm{A}\}$ is a directed family in X, the ranges of the sections of this family form a filter base on X; the corresponding filter is called the **section filter** of the family.

An **elementary** filter is the section filter of a sequence $\{x_n : n \in N\}$ in X (N being endowed with its usual order).

Literature. Sets: Bourbaki [1], Halmos [3]. Filters: Bourbaki [4], Bushaw [1]. Order: Birkhoff [1], Bourbaki [1].

B. GENERAL TOPOLOGY

1. *Topologies.* Let X be a set, $\mathfrak{G}$ a set of subsets of X invariant under finite intersections and arbitrary unions; it follows that $X \in \mathfrak{G}$, since X is the intersection of the empty subset of $\mathfrak{G}$, and that $\varnothing \in \mathfrak{G}$, since $\varnothing$ is the union of the empty subset of $\mathfrak{G}$. We say that $\mathfrak{G}$ defines a **topology** $\mathfrak{T}$ on X; structurized in this way, X is called a **topological space** and denoted by $(X, \mathfrak{T})$ if reference to $\mathfrak{T}$ is desirable. The sets $G \in \mathfrak{G}$ are called **open**, their complements $F = X \sim G$ are called **closed** (with respect to $\mathfrak{T}$). Given $A \subset X$, the open set $\mathring{A}$ (or int A) which is the union of all open subsets of A, is called the **interior** of A; the closed set $\bar{A}$, intersection of all closed sets containing A, is called the **closure** of A. An element $x \in \mathring{A}$ is called an **interior point** of A (or interior to A), an element $x \in \bar{A}$ is called a **contact point (adherent point)** of A. If A, B are subsets of X, B is **dense** relative to A if $A \subset \bar{B}$ (**dense** *in* A if $B \subset A$ and $A \subset \bar{B}$). A topological space X is **separable** if X contains a countable dense subset; X is **connected** if X is not the union of two disjoint non-empty open subsets (otherwise, X is **disconnected**).

Let X be a topological space. A subset $U \subset X$ is a **neighborhood** of x if $x \in \mathring{U}$, and a neighborhood of A if $x \in A$ implies $x \in \mathring{U}$. The set of all neighborhoods of x (respectively, of A) is a filter on X called the **neighborhood filter** of x (respectively, of A); each base of this filter is a **neighborhood base** of x (respectively, of A). A bijection f of X onto another topological space Y such that $f(A)$ is open in Y if and only if A is open in X, is called a **homeomorphism**; X and Y are **homeomorphic** if there exists a homeomorphism of X onto Y. The **discrete topology** on X is the topology for which every subset of X is open; the **trivial** topology on X is the topology whose only open sets are $\varnothing$ and X.

2. *Continuity and Convergence.* Let X, Y be topological spaces and let $f: X \to Y$. f is **continuous at** $x \in X$ if for each neighborhood V of $y = f(x)$, $f^{-1}(V)$ is a neighborhood of x (equivalently, if the filter on Y generated by the base $f(\mathfrak{U})$ is finer than $\mathfrak{B}$, where $\mathfrak{U}$ is the neighborhood filter of x, $\mathfrak{B}$ the neighborhood filter of y). f is **continuous** on X into Y (briefly, continuous) if f is continuous at each $x \in X$ (equivalently, if $f^{-1}(G)$ is open in X for each open $G \subset Y$). If Z is also a topological space and $f: X \to Y$ and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous.

A filter $\mathfrak{F}$ on a topological space X is said to **converge** to $x \in X$ if $\mathfrak{F}$ is finer than the neighborhood filter of x. A sequence (more generally, a directed family) in X **converges** to $x \in X$ if its section filter converges to x. If also Y

is a topological space and $\mathfrak{F}$ is a filter (or merely a filter base) on X, and if $f: X \to Y$ is a map, then f is said to **converge** to $y \in Y$ *along* $\mathfrak{F}$ if the filter generated by $f(\mathfrak{F})$ converges to y. For example, f is continuous at $x \in X$ if and only if f converges to $y = f(x)$ along the neighborhood filter of x. Given a filter $\mathfrak{F}$ on X and $x \in X$, x is a **cluster point (contact point, adherent point)** of $\mathfrak{F}$ if $x \in \bar{F}$ for each $F \in \mathfrak{F}$. A **cluster point** of a sequence (more generally, of a directed family) is a cluster point of the section filter of this family.

3. *Comparison of Topologies.* If X is a set and $\mathfrak{T}_1, \mathfrak{T}_2$ are topologies on X, we say that $\mathfrak{T}_2$ is **finer** than $\mathfrak{T}_1$ (or $\mathfrak{T}_1$ **coarser** than $\mathfrak{T}_2$) if every $\mathfrak{T}_1$-open set is $\mathfrak{T}_2$-open (equivalently, if every $\mathfrak{T}_1$-closed set is $\mathfrak{T}_2$-closed). (If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are the respective families of open sets in X, this amounts to the relation $\mathfrak{G}_1 \subset \mathfrak{G}_2$ in $\mathfrak{P}(\mathfrak{P}(X))$.) Let $\{\mathfrak{T}_\alpha: \alpha \in A\}$ be a family of topologies on X. There exists a finest topology $\mathfrak{T}$ on X which is coarser than each $\mathfrak{T}_\alpha (\alpha \in A)$; a set G is $\mathfrak{T}$-open if and only if G is $\mathfrak{T}_\alpha$-open for each α. Dually, there exists a coarsest topology $\mathfrak{T}_0$ which is finer than each $\mathfrak{T}_\alpha (\alpha \in A)$. If we denote by $\mathfrak{G}_0'$ the set of all finite intersections of sets open for some $\mathfrak{T}_\alpha$, the set $\mathfrak{G}_0$ of all unions of sets in $\mathfrak{G}_0'$ constitutes the $\mathfrak{T}_0$-open sets in X. Hence with respect to the relation "$\mathfrak{T}_2$ is finer than $\mathfrak{T}_1$", the set of all topologies on X is a complete lattice; the coarsest topology on X is the trivial topology, the finest topology is the discrete topology. The topology $\mathfrak{T}$ is the greatest lower bound (briefly, *the* lower bound) of the family $\{\mathfrak{T}_\alpha: \alpha \in A\}$; similarly, $\mathfrak{T}_0$ is *the* upper bound of the family $\{\mathfrak{T}_\alpha: \alpha \in A\}$.

One derives from this two general methods of defining a topology (Bourbaki [4]). Let X be a set, $\{X_\alpha: \alpha \in A\}$ a family of topological spaces. If $\{f_\alpha: \alpha \in A\}$ is a family of mappings, respectively of X into X_α, the **projective topology (kernel topology)** on X with respect to the family $\{(X_\alpha, f_\alpha): \alpha \in A\}$ is the coarsest topology for which each f_α is continuous. Dually, if $\{g_\alpha: \alpha \in A\}$ is a family of mappings, respectively of X_α into X, the **inductive topology (hull topology)** with respect to the family $\{(X_\alpha, g_\alpha): \alpha \in A\}$ is the finest topology on X for which each g_α is continuous. (Note that each f_α is continuous for the discrete topology on X, and that each g_α is continuous for the trivial topology on X.) If $A = \{1\}$ and $\mathfrak{T}_1$ is the topology of X_1, the projective topology on X with respect to (X_1, f_1) is called the **inverse image** of $\mathfrak{T}_1$ under f_1, and the inductive topology with respect to (X_1, g_1) is called the **direct image** of $\mathfrak{T}_1$ under g_1.

4. *Subspaces, Products, Quotients.* If $(X, \mathfrak{T})$ is a topological space, A a subset of X, f the canonical imbedding $A \to X$, then the **induced** topology on A is the inverse image of $\mathfrak{T}$ under f. (The open subsets of this topology are the intersections with A of the open subsets of X.) Under the induced topology, A is called a **topological subspace** of X (in general, we shall avoid this terminology because of possible confusion with vector subspaces). If $(X, \mathfrak{T})$ is a topological space, R an equivalence relation on X, g the canonical map $X \to X/R$, then the direct image of $\mathfrak{T}$ under g is called the **quotient** (topology) of $\mathfrak{T}$; under this topology, X/R is the **topological quotient** of X by R.

Let $\{X_\alpha: \alpha \in A\}$ be a family of topological spaces, X their Cartesian product, f_α the projection of X onto X_α. The projective topology on X with respect to the family $\{(X_\alpha, f_\alpha): \alpha \in A\}$ is called the **product topology** on X. Under this topology, X is called the **topological product** (briefly, **product**) of the family $\{X_\alpha: \alpha \in A\}$.

Let X, Y be topological spaces, f a mapping of X into Y. We say that f is **open** (or an **open map**) if for each open set $G \subset X, f(G)$ is open in the topological subspace $f(X)$ of Y. f is called **closed** (a **closed map**) if the graph of f is a closed subset of the topological product $X \times Y$.

5. *Separation Axioms.* Let X be a topological space. X is a **Hausdorff** (or **separated**) space if for each pair of distinct points x,y there are respective neighborhoods U_x, U_y such that $U_x \cap U_y = \varnothing$. If (and only if) X is separated, each filter $\mathfrak{F}$ that converges in X, converges to exactly one $x \in X$; x is called the **limit** of $\mathfrak{F}$. X is called **regular** if it is separated and each point possesses a base of closed neighborhoods; X is called **normal** if it is separated and for each pair A, B of disjoint closed subsets of X, there exists a neighborhood U of A and a neighborhood V of B such that $U \cap V = \varnothing$.

A Hausdorff topological space X is normal if and only if for each pair A, B of disjoint closed subsets of X, there exists a continuous function f on X into the real interval $[0,1]$ (under its usual topology) such that $f(x) = 0$ whenever $x \in A, f(x) = 1$ whenever $x \in B$ (Urysohn's theorem).

A separated space X such that for each closed subset A and each $b \notin A$, there exists a continuous function $f\colon X \to [0,1]$ for which $f(b) = 1$ and $f(x) = 0$ whenever $x \in A$, is called **completely regular**; clearly, every normal space is completely regular, and every completely regular space is regular.

6. *Uniform Spaces.* Let X be a set. For arbitrary subsets W, V of $X \times X$, we write $W^{-1} = \{(y, x): (x, y) \in W\}$, and $V \circ W = \{(x, z):$ *there exists* $y \in X$ *such that* $(x, y) \in W, (y, z) \in V\}$. The set $\Delta = \{(x, x): x \in X\}$ is called the **diagonal** of $X \times X$. Let $\mathfrak{W}$ be a filter on $X \times X$ satisfying these axioms:

(1) *Each* $W \in \mathfrak{W}$ *contains the diagonal* Δ.
(2) $W \in \mathfrak{W}$ *implies* $W^{-1} \in \mathfrak{W}$.
(3) *For each* $W \in \mathfrak{W}$, *there exists* $V \in \mathfrak{W}$ *such that* $V \circ V \subset W$.

We say that the filter $\mathfrak{W}$ (or any one of its bases) defines a **uniformity** (or **uniform structure**) on X, each $W \in \mathfrak{W}$ being called a **vicinity (entourage)** of the uniformity. Let $\mathfrak{G}$ be the family of all subsets G of X such that $x \in G$ implies the existence of $W \in \mathfrak{W}$ satisfying $\{y: (x, y) \in W\} \subset G$. Then $\mathfrak{G}$ is invariant under finite intersections and arbitrary unions, and hence defines a topology $\mathfrak{T}$ on X such that for each $x \in X$, the family $W(x) = \{y: (x, y) \in W\}$, where W runs through $\mathfrak{W}$, is a neighborhood base of x. The space $(X, \mathfrak{W})$, endowed with the topology $\mathfrak{T}$ derived from the uniformity $\mathfrak{W}$, is called a **uniform space**. A topological space X is **uniformisable** if its topology can be derived from a uniformity on X; the reader should be cautioned that, in general, such a uniformity is not unique.

A uniformity is **separated** if its vicinity filter satisfies the additional axiom

(4) $\bigcap \{W: W \in \mathfrak{W}\} = \Delta$.

(4) is a necessary and sufficient condition for the topology derived from the uniformity to be a Hausdorff topology. A Hausdorff topological space is uniformisable if and only if it is completely regular.

Let X, Y be uniform spaces. A mapping $f: X \to Y$ is **uniformly continuous** if for each vicinity V of Y, there exists a vicinity U of X such that $(x,y) \in U$ implies $(f(x), f(y)) \in V$. Each uniformly continuous map is continuous. The uniform spaces X, Y are **isomorphic** if there exists a bijection f of X onto Y such that both f and f^{-1} are uniformly continuous; f itself is called a **uniform isomorphism**.

If $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two filters on $X \times X$, each defining a uniformity on the set X, and if $\mathfrak{W}_1 \subset \mathfrak{W}_2$, we say that the uniformity defined by $\mathfrak{W}_1$ is **coarser** than that defined by $\mathfrak{W}_2$. If X is a set, $\{X_\alpha: \alpha \in \mathrm{A}\}$ a family of uniform spaces and $f_\alpha(\alpha \in \mathrm{A})$ are mappings of X into X_α, then there exists a coarsest uniformity on X for which each $f_\alpha(\alpha \in \mathrm{A})$ is uniformly continuous. In this way, one defines the **product uniformity** on $X = \prod_\alpha X_\alpha$ to be the coarsest uniformity for which each of the projections $X \to X_\alpha$ is uniformly continuous; similarly, if X is a uniform space and $A \subset X$, the **induced uniformity** is the coarsest uniformity on A for which the canonical imbedding $A \to X$ is uniformly continuous.

Let X be a uniform space. A filter $\mathfrak{F}$ on X is a **Cauchy filter** if, for each vicinity V, there exists $F \in \mathfrak{F}$ such that $F \times F \subset V$. If each Cauchy filter converges (to an element of X) then X is called **complete**. To each uniform space X one can construct a complete uniform space $\tilde{X}$ such that X is (uniformly) isomorphic with a dense subspace of $\tilde{X}$, and such that $\tilde{X}$ is separated if X is. If X is separated, then $\tilde{X}$ is determined by these properties to within isomorphism, and is called the **completion** of X. A base of the vicinity filter of $\tilde{X}$ can be obtained by taking the closures (in the topological product $\tilde{X} \times \tilde{X}$) of a base of vicinities of X. A **Cauchy sequence** in X is a sequence whose section filter is a Cauchy filter; if every Cauchy sequence in X converges, then X is said to be **semi-complete (sequentially complete)**.

If X is a complete uniform space and A a closed subspace, then the uniform space A is complete; if X is a separated uniform space and A a complete subspace, then A is closed in X. A product of uniform spaces is complete if and only if each factor space is complete.

If X is a uniform space, Y a complete separated space, $X_0 \subset X$ and $f: X_0 \to Y$ uniformly continuous; then f has a unique uniformly continuous extension $\bar{f}: \bar{X}_0 \to Y$.

7. *Metric and Metrizable Spaces.* If X is a set, a non-negative real function d on $X \times X$ is called a **metric** if the following axioms are satisfied:

(1) $d(x, y) = 0$ *is equivalent with* $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \leqq d(x, y) + d(y, z)$ (*triangle inequality*).

Clearly, the sets $W_n = \{(x, y): d(x, y) < n^{-1}\}$, where $n \in N$, form a filter base on $X \times X$ defining a separated uniformity on X; by the **metric space** (X, d) we understand the uniform space X endowed with the metric d. Thus all uniform concepts apply to metric spaces. (It should be understood that, historically, uniform spaces are the upshot of metric spaces.) A topological space is **metrizable** if its topology can be derived from a metric in the manner indicated; a uniform space is metrizable (i.e., its uniformity can be generated by a metric) if and only if it is separated and its vicinity filter has a countable base. Clearly, a metrizable uniform space is complete if it is semi-complete.

8. *Compact and Precompact Spaces.* Let X be a Hausdorff topological space. X is called **compact** if every open cover of X has a finite subcover. For X to be compact, each of the following conditions is necessary and sufficient: (a) A family of closed subsets of X has non-empty intersection whenever each finite subfamily has non-empty intersection. (b) Each filter on X has a cluster point. (c) Each ultrafilter on X converges.

Every closed subspace of a compact space is compact. The topological product of any family of compact spaces is compact (Tychonov's theorem). If X is compact, Y a Hausdorff space, and $f: X \to Y$ continuous, then $f(X)$ is a compact subspace of Y. If f is a continuous bijection of a compact space X onto a Hausdorff space Y, then f is a homeomorphism (equivalently: If $(X, \mathfrak{T}_1)$ is compact and $\mathfrak{T}_2$ is a Hausdorff topology on X coarser than $\mathfrak{T}_1$, then $\mathfrak{T}_1 = \mathfrak{T}_2$).

There is the following important relationship between compactness and uniformities: On every compact space X, there exists a unique uniformity generating the topology of X; the vicinity filter of this uniformity is the neighborhood filter of the diagonal Δ in the topological product $X \times X$. In particular, every compact space is a complete uniform space. A separated uniform space is called **precompact** if its completion is compact. (However, note that a topological space can be precompact for several distinct uniformities yielding its topology.) X is precompact if and only if for each vicinity W, there exists a finite subset $X_0 \subset X$ such that $X \subset \bigcup \{W(x): x \in X_0\}$. A subspace of a precompact space is precompact, and the product of any family of precompact spaces is precompact.

A Hausdorff topological space is called **locally compact** if each of its points possesses a compact neighborhood.

9. *Category and Baire Spaces.* Let X be a topological space, A a subset of X. A is called **nowhere dense (rare)** in X if its closure $\bar{A}$ has empty interior; A is called **meager** (of **first category**) in X if A is the union of a countable set of rare subsets of X. A subset A which is not meager is called **non-meager** (of **second category**) in X; if every non-empty open subset is nonmeager in X, then X is called a **Baire space**. Every locally compact space and every complete

metrizable space is a Baire space (Baire's theorem). Each non-meager subset of a topological space X is non-meager in itself, but a topological subspace of X can be a Baire space while being a rare subset of X.

Literature: Berge [1]; Bourbaki [4], [5], [6]; Kelley [1]. A highly recommendable introduction to topological and uniform spaces can be found in Bushaw [1].

C. LINEAR ALGEBRA

1. *Vector Spaces.* Let L be a set, K a (not necessarily commutative) field. Suppose there are defined a mapping $(x, y) \to x + y$ of $L \times L$ into L, called **addition**, and a mapping $(\lambda, x) \to \lambda x$ of $K \times L$ into L, called **scalar multiplication**, such that the following axioms are satisfied (x, y, z denoting arbitrary elements of L, and λ, μ arbitrary elements of K):

(1) $(x + y) + z = x + (y + z)$.
(2) $x + y = y + x$.
(3) *There exists an element* $0 \in L$ *such that* $x + 0 = x$ *for all* $x \in L$.
(4) *For each* $x \in L$, *there exists* $z \in L$ *such that* $x + z = 0$.
(5) $\lambda(x + y) = \lambda x + \lambda y$.
(6) $(\lambda + \mu)x = \lambda x + \mu x$.
(7) $\lambda(\mu x) = (\lambda\mu)x$.
(8) $1x = x$.

Endowed with the structure so defined, L is called a **left vector space** over K. The element 0 postulated by (3) is unique and called the **zero element** of L. (We shall not distinguish notationally between the zero elements of L and K.) Also, for any $x \in L$ the element z postulated by (4) is unique and denoted by $-x$; moreover, one has $-x = (-1)x$, and it is customary to write $x - y$ for $x + (-y)$.

If (1)–(4) hold as before but scalar multiplication is written $(\lambda, x) \to x\lambda$ and (5)–(8) are changed accordingly, L is called a **right vector space** over K. By a **vector space over** K, we shall always understand a left vector space over K. Since there is no point in distinguishing between left and right vector spaces over K when K is commutative, there will be no need to consider right vector spaces except in C.4 below, and Chapter I, Section 4. (From Chapter II on, K is always supposed to be the real field $\boldsymbol{R}$ or the complex field $\boldsymbol{C}$.)

2. *Linear Independence.* Let L be a vector space over K. An element $\lambda_1 x_1 + \cdots + \lambda_n x_n$, where $n \in N$, is called a **linear combination** of the elements $x_i \in L (i = 1, \ldots, n)$; as usual, this is written $\sum_{i=1}^{n} \lambda_i x_i$ or $\sum_i \lambda_i x_i$. If $\{x_\alpha : \alpha \in H\}$ is a finite family, the sum of the elements x_α is denoted by $\sum_{\alpha \in H} x_\alpha$; for reasons of convenience, this is extended to the empty set by defining $\sum_{x \in \emptyset} x = 0$. (This