Edward A. Silver, *Editor*

Teaching and Learning Mathematical Problem Solving:

Multiple Research Perspectives

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Introduction

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In recent years, considerable progress has been made in understanding the nature of complex human thought, with particular insights into the mechanisms of skilled problem solving. Much of the early research on human problem solving concentrated on general problem-solving skills and used puzzle problems and highly structured tasks with a dearth of semantic content, such as the "Towers of Hanoi" or "Missionaries and Cannibals" problems. More recent research, however, has focused on problem-solving behavior in semantically rich knowledge domains with direct or indirect relevance to mathematics. The participants in this research enterprise have come from many fields, but primarily from mathematics education, cognitive psychology, science education, and artificial intelligence.

Corresponding to the intense level of research attention to the mechanisms of mathematical problem solving has been a growing interest in the topic among educational practitioners. Since the publication by the National Council of Teachers of Mathematics of the *Agenda for Action,* which asserted that the acquisition of problem-solving skills should be one of the goals of school mathematics instruction in the 1980s, problem solving has been a dominant topic at virtually all professional meetings of mathematics teachers and supervisors. Rarely in the history of education has a topic simultaneously captured so much of the attention of both researchers and practitioners. Usually, the research community is busy investigating a topic long after it ceases to be of real interest to practitioners.

Another interesting trend—a focus on the processes of learning—has made the time ripe for fruitful contributions of research on human cognition to classroom practice. Although some early workers in artificial intelligence were apparently quite interested in learning, it is only in recent years that much attention has been given to creating systems that learn. Similarly, cognitive psychologists who had heretofore focused their attention on the issue of performance, perhaps as a negative reaction to the obvious interest of behaviorist psychologists in learning (e.g., operant conditioning, maze learning), have re-discovered learning as an issue of import and interest. In recent years, a great deal of enthusiasm for tackling the mechanisms of learning has appeared in the cognitive science community. Thus, the moment seems opportune for the mathematics education and cognitive science communities to benefit from one another.

Despite the fact that a considerable amount of research on the topic of mathematical problem solving has been amassed, and despite the widespread interest on the part of practitioners, there have been few attempts to synthesize that research base and even fewer attempts to synthesize the research from the perspective of educational practice. This observation led me to propose to the National Science Foundation that the time was ripe for an assessment of the state-of-the-art in research on mathematical problem solving. The proposed synthesis was to have particular emphasis on classroom practice and on increased communication across the disciplines that have been active in this research.

It was in that spirit, and as a result of that proposal, that almost 40 researchers and practitioners interested in mathematical problem solving gathered at San Diego State University in June 1983. Many of the participants at the conference were consultants to the NSF project who had prepared papers that reported on new developments in their particular fields of expertise. The conference on "Teaching Mathematical Problem Solving: Multiple Research Perspectives" had three specific aims: (*1*) to provide a current assessment of the state-of-the-art in research on mathematical problem solving from a mathematics education perspective; (*2*) to explore the potential contributions of other research and other perspectives from fields such as cognitive psychology, scientific problem solving, and artificial intelligence; and (*3*) to identify some productive directions for research on mathematical problem solving in the next decade. Although the conference agenda was quite broad, it succeeded in all its aims. This book contains the set of papers that were read by the participants in advance, discussed at the meeting, and subsequently revised by the authors to reflect participants' comments.

In keeping with the aims of the conference, this volume is organized into sections corresponding to the three major aims of the meeting. Part A includes four papers written by mathematics educators who survey the field broadly and provide summaries of major bodies of problem-solving research or research issues. Part B includes four pairs of papers presenting the contrasting perspectives of researchers who work in different fields —mathematics education, cognitive psychology, scientific problem solving, or artificial intelligence. Part C includes 11 papers that point the way toward potentially fruitful directions for future research on mathematical problem solving. Many of the papers in Part C especially embody the spirit of the conference, since they represent perspectives that are compatible with research in both cognitive science and mathematics education. The volume concludes, as the conference did, with a summary paper by Lee Shulman, who presents his own unique perspective concerning research on mathematical problem solving.

Part A opens with "A Retrospective Account of the Past Twenty-five Years of Research on Teaching Mathematical Problem Solving," in which Jeremy Kilpatrick reviews and summarizes many of the major issues that have occupied research attention in the field during the past two and one-half decades. In the next paper, Tom Carpenter discusses and summarizes the voluminous literature on young children's abilities to solve addition and subtraction story problems. Carpenter's paper is one of the first serious attempts to relate this area of problem-solving research to the rest of the field. Important methodological issues are the focus of Frank Lester's paper. He discusses the changes in methodology that have occurred in problem-solving research over the past several decades and gives serious attention to the problems that those changes have created as well as to the problems that still remain from previous research methodologies. In the final paper in this section, Nick Branca, fresh from his travel in Great Britain, discusses the British perspective on problem solving and draws lessons from his experience in England.

Cross-discipline perspectives are presented in Part B. This section contains four pairs of papers; one paper in each pair summarizes important research findings or perspectives from one discipline, and the second paper presents the reaction of a person representing another discipline. Edwina Rissland presents a useful summary of much of the work in artificial intelligence, and Alan Schoenfeld, a mathematician whose own research has carried him into the realm of cognitive science, provides a reaction from the perspective of mathematics education. In his paper dealing with research on cognitive psychology, Richard Mayer surveys some of the work that is relevant to standard textbook story problems; Larry Sowder, whose own recent research has dealt with story problems, provides his reaction from a mathematics education perspective. In the area of scientific problem solving, Joan Heller and Harriet Hungate provide an excellent and detailed summary of some of the interesting research dealing with problem solving in physics; Gerald Goldin, whose research interests include theoretical physics as well as mathematics education, provides the reaction. The fourth pair of papers consists of Patrick Thompson's views on reasonable research perspectives to guide curriculum design, especially in developing computer microworlds, and Jim Greeno's reaction from the perspective of a cognitive psychologist. Thompson's paper makes vivid the contrast between researchers who study only what *is* and those who also study what *ought to be* in the curriculum.

Part C is introduced with my paper, "Research on Teaching Mathematical Problem Solving: Some Underrepresented Themes and Needed Directions," in which I discuss a number of issues and themes that have emerged from review and synthesis of the mathematical problem-solving literature. Subsequent papers discuss in more detail the themes and issues developed in my paper. The fundamental importance of epistemological issues (e.g., beliefs, conceptions, misconceptions) is reflected in the papers by Jim Kaput, Richard Lesh, Alan Schoenfeld, and Mike Shaughnessy. Another common theme is the need for the careful study of classroom processes and teachers and their role in problem-solving instruction. That theme is reflected in the papers by Alba Thompson, Doug Grouws, and Nel Noddings. The importance of affective issues in the study of problem-solving performance, an oft-forgotten theme in problem-solving research, is discussed by Doug McLeod and Nel Noddings. In other papers, Judith Threadgill-Sowder addresses the role of individual cognitive differences in mathematical problem solving, and Patrick Thompson examines some of the potential impact that computers may have in research on mathematical problem solving. Taken collectively, this set of papers constitutes an ambitious research agenda for the next decade.

Those who attended the conference reported a great deal of enthusiasm about the experience. Although it is not possible to capture on paper all of what occurred at the meeting, this volume represents an accurate rendering of the substance of the conference. I hope that readers of this book will experience some of the excitement that the conference participants felt for the ideas expressed in these papers. If this volume leads to an increased understanding of the nature of past problem-solving research, or an increased appreciation of the potential benefits of cross-fertilization of ideas among workers in different fields, or a renewed enthusiasm for attacking some of the underrepresented themes and issues from previous research, then it will have served its purpose. I look forward eagerly to a future volume on this topic, in which the specific results of this next wave of research could be presented and discussed.

PART A.
PROBLEM-SOLVING RESEARCH: MATHEMATICS EDUCATION PERSPECTIVES

Contents

Part C. Problem-Solving Research: Themes and Directions

Summary

A Retrospective Account of the Past 25 Years of Research on Teaching Mathematical Problem Solving

Jeremy Kilpatrick
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I readily accepted the assignment posed by the title of this paper because I was interested in where it might lead. It is a little like a grown-up version of "what I did on my summer vacation"—the challenge is to discover what you can make of someone else's attempt to draw you out on a theme.

In this case, my first thoughts concerned the choice of 25 years as a time span for this retrospective account. What were you doing 25 years ago? Were you interested in problem solving in mathematics? If I can wax autobiographical for a bit, 25 years ago this month I was finishing up my first year as a junior high school teacher of mathematics and science in the Berkeley, California, public schools. I was also working part-time on a master's degree in education at the University of California. Despite my greenhorn status, I had been selected to attend an institute for mathematics teachers at Stanford University that was being offered during that summer of 1958. The institute was sponsored by General Electric and was one of the forerunners of the National Science Foundation institutes. I was about to overcome my natural aversion to crossing the San Francisco Bay to Palo Alto because I wanted to learn more about mathematics teaching from people like Harold Bacon, Ivan Niven, and Morris Kline. I also wanted to see what Stanford was like as a place for graduate study in mathematics education. I was hoping that, if I went to Stanford, I might have some opportunity to get to know George Polya and his ideas better.

I had run across Polya before—as a high school student I had purchased a copy of *How to Solve It* and, although I couldn't always see the point of Polya's remarks, it was obvious that he was on to something. As a senior in high school, I took the Stanford Mathematics Examination — which I later learned had been created and conducted by Polya and Gabor Szego — and was dismayed to discover the depth of my ignorance of mathematics; I had never encountered mathematical problems before that required more than 10 minutes' thought, and here were three problems to be solved in 3 hours.

Later, when I was an undergraduate mathematics major, I sat in on some lectures that Polya, who was visiting at Berkeley, was offering for liberal arts students. The lectures gave me a clearer impression of what he meant by problem solving and how that might be an important thing for mathematics teachers to know about.

In June 1958, after a year of teaching first-year algebra, I could see that the solution of word problems was posing special pedagogical difficulties. Despite my lucid explanations of how to translate from English to algebra, my firm insistence that students write out their assignment of letters to variables in the problem, my helpful suggestions that they use tables to organize the variables and diagrams to represent relationships, and my rigid requirement that the last part of the solution be an English sentence that responded to the question posed in the problem statement, my students nonetheless moaned with pain when the word-problem sections of our textbook rolled around and stared with incomprehension and chagrin when we discussed the word problems they had failed to solve on their homework or a test. Conferences with parents were all too likely to revolve around the parents' claim that Barbara or Steve, though blessed with the genius of a von Neumann when it came to solving equations or simplifying fractions, could make no sense whatsoever of those three-line epigrams about mixing peanuts and cashews. "Problem solving," in the guise of the hackneyed word problems in Mallory's *First Algebra*, had become my Tar Baby, and a quarter of a century later I am still wrestling with it.

Why am I telling you this? Partly, perhaps, because the author of a keynote paper ought to be entitled to a little self-indulgence. But primarily because I would like to trace for you some strands in the research done on the teaching of mathematical problem solving over the last 25 years, and I want to use my situation as a mathematics teacher in June 1958 as an anchoring point for these strands.

WHAT IS A PROBLEM?

One strand concerns our view of what a mathematical problem is. This view can be general—in which we look at a problem and the process of solving it from different perspectives—or it can be specifically focused on the roles that problems play in mathematics teaching. Let us consider each in turn.

Perspectives on Mathematical Problems

In what might be termed the *psychological* perspective on problems, a problem is defined generally as a situation in which a goal is to be attained and a direct route to the goal is blocked. The psychologists usually add that the situation requires the presence of a person who "has" the problem; to be a problem, it has to be a problem for someone. (The psychologists are not usually concerned with mathematical problems per se, but one could add to the formulation that for a problem to be mathematical, mathematical concepts and principles should be used in seeking the answer.) From this perspective, we see the problem as an *activity* of a motivated subject.

The subjective nature of a mathematical problem, although often overlooked in discussions of problem solving, has been understood and accepted in mathematics education for at least 40 years. In the autumn of 1957, I had taken a psychological foundations course from William Brownell, who was the Dean of the College of Education at Berkeley, and in the spring of 1958, I was taking a course from him on research in the teaching of arithmetic. I had read his classic paper on problem solving (Brownell, 1942) and was familiar with his definition:

> Problem solving refers (a) only to perceptual and conceptual tasks, (b) the nature of which the subject by reason of original nature, of previous learning, or of organization of the task, is able to understand, but (c) for which at the time he knows no direct means of satisfaction. (d) The subject experiences perplexity in the problem situation, but he does not experience utter confusion. . . . Problem solving becomes the process by which the subject extricates himself from his problem. . . . Defined thus, problems may be thought of as occupying intermediate territory in a continuum which stretches from the 'puzzle' at one extreme to the completely familiar and understandable situation at the other. (p. 416)

Researchers in mathematics education have long accepted the truth that a problem for you today may not be one for me today or for you tomorrow.

Something the last 25 years have given us, however, is a greater appreciation of the mathematical problem as *task*—a *social-anthropological* perspective. Namely, the problem is given and received in a transaction. The mathematics classroom is a social situation jointly constructed by the participants, in which teacher and students interpret each other's actions and intentions in the light of their own agendas (Mehan, 1978).

> The teacher ordinarily assigns a problem to one or more students, who are then expected to solve it. Students who are assigned a problem in class can bring to bear on the task a whole set of considerations that would not be operative if the students had formulated and posed the problems on their own: For example, they can usually assume that the problem will have a single, well-defined answer obtainable using procedures they have studied recently in class; they can assume that they will be evaluated by the teacher according to how hard they appear to try in solving the problem and how successful they appear to be; they may have some indication that the teacher knows the solution to the problem, and they may be able to get clues to the solution by asking the teacher certain questions; and they may be able to use gestural and postural cues from the teacher to tell whether or not they are on the right track in their solution. (Kilpatrick, 1982, pp. 10–11)

Researchers in mathematics education are just beginning to examine the implications of the social-anthropological point of view (Brandau, 1981).

Other views include the *mathematical* (problem as *construction*) and the *pedagogical* (problem as *vehicle*). From the mathematical, or subject-matter, point of view, one sees mathematical problems as defining the discipline of mathematics. Mathematics is not simply the famous problems that great mathematicians have worked on; "*all* mathematics is created in the process of formulating and solving problems" (Kilpatrick, 1982, p. 2). Schoolchildren seldom see much of this face of mathematics; for them, the construction scaffolding has been taken away, and all that remains is the completed edifice through which they are guided. Although the subject-matter point of view has some important implications for research, I want to consider at length instead the pedagogical, or curricular, point of view—what a mathematical problem means in teaching—as a way of accounting for some of the variety to be seen in the research on teaching mathematical problem solving.

The Roles of Mathematical Problems in Teaching

Word problems (sometimes termed verbal problems or story problems) were my own point of entry into the problem-solving domain, and they serve to define the territory for many researchers. Polya (1981, Vol. 2, p. 139) has given us a nice classification of problems from a pedagogical perspective:

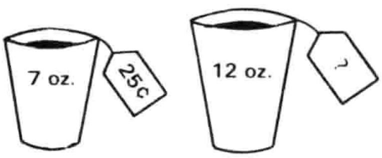
1. *One rule under your nose*—the type of problem to be solved by mechanical application of a rule that has just been presented or discussed.
2. *Application with some choice*—a problem that can be solved by application of a rule or procedure given earlier in class so that the solver has to use some judgment.
3. *Choice of a combination*—a problem that requires the solver to combine two or more rules or examples given in class.
4. *Approaching research level*—a problem that also requires a novel combination of rules or examples but that has many ramifications and requires a high degree of independence and the use of plausible reasoning.

Polya argues that both the degree of difficulty and the educational value (with respect to teaching students to think) increase as one goes from type 1 to type 4. Most research over the past 25 years has concentrated on problems of types 1 and 2, but increasing attention is being given by some researchers in mathematics education to type 3 and 4 problems.

I would like to offer a somewhat different view that is meant to encompass both in-school and out-of-school mathematical problems. Consider the following six problems. They are chosen only to illustrate various meanings of "problem"; in no sense are they meant to be typical or estimable.

a. $^{7}/_{25} = {}^{12}/_{x}$. What is x?

b.



c. If a 7-oz. cup of cola costs 25¢, what is the cost of a 12-oz. cup?

d. A group of three sixth-graders has been given the problem of planning a picnic. They have been told that 7-oz. cups of cola cost 25¢ each, and they are trying to find out what 12-oz. cups should cost.

e. Your neighborhood association is having a picnic and is hoping to make some money selling cups of cola. One of the officers has set a price of 25¢ for the 7-oz. cups and has asked you what price would be fair for the 12-oz. cups.

f. If a 7-oz. cup of cola costs 25¢, the proportional cost of a 12-oz. cup is not a whole number. What is the smallest whole number one could add to the cost of the 7-oz. cup to make the proportional cost of the 12-oz. cup a whole number?

Problem *a* might be termed the computational skeleton beneath the skin of the other problems, although there are, of course, many other ways to pose the computational question. Problems *b* and *c* are straightforward "word problems" of the type in which seventh- and eighth-grade textbooks abound. They are meant to give students an opportunity to apply what they might have learned—in this case, what they might have learned about proportion. Problem *b* is something of a new breed: the "wordless" word problem. Such problems can be seen in recent textbooks whose authors are attempting to cope with the difficulties students have in reading word problems or other material with understanding. Problem *c* is the word problem in its more traditional garb.

Problem *d* is meant to exemplify a class of problems sometimes termed "real problems." These are problems that students either pose for themselves or are given, in which the students are in a "realistic" situation that is meaningful to them and must come up with a solution that takes account of the problem setting as well as any mathematics that might be appropriate. The problem is meant to simulate a real problem the students might face. Problem *e*, on the other hand, is meant to exemplify the class of problems—potentially involving some mathematics—that students might encounter outside the school setting, when scores, grades, and research results are not at stake. Problem *f* represents, in a very limited fashion, the class of nonroutine mathematical problems of greater complexity and greater mathematical interest than the ordinary word problem—in this instance, Problem *f* might lead to a consideration of Diophantine equations in two unknowns.

Which of these kinds of problems would we like students to be able to solve? Some people might hold that it is problems like *e* and *f* that we are ultimately interested in—Problem *e* for everyone; Problem *f* for the more capable in mathematics, perhaps—and the other problems are the ones we should be concerned with in school. Some might say that if we teach students how to solve problems of type *a* and type *b* or *c*, everything else will take care of itself. Others might want to substitute problems of type *d* or *f* for the masses of type-*c* problems in the textbooks, arguing that the type-*c* problems are a case of bad currency driving out good. The point is that different people hold different views concerning the kinds of problems that ought to be dealt with in the mathematics curriculum and how much weight each kind of problem should be given. These different views have not always been distinguished by researchers concerned with problem solving, nor have they explored the consequences of these views in any detail.

The media have given considerable attention to the results of the Second Mathematics Assessment of the National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981), which suggest that American schoolchildren are a lot better at solving type-*a* problems than type-*c* problems. One needs to be somewhat cautious in drawing inferences from this finding, however. First, we have no evidence to suggest that there was a time when students solved both types of problems with equal facility. Second, and more important, a failure to solve type-*c* problems does not necessarily imply that students, in their lives outside school, will not be able to give satisfactory solutions to type-*e* problems. Much has been made of the implications of the poor showing students have made on type-*c* problems. But where is it written—indeed, where has it been shown—that facility with type-*c* problems is a prerequisite for intelligent problem solving outside of school? The evidence we have suggests that children often fail to use the standard methods they have been taught when type-*c* problems are presented out of context (Hart, 1983); how much more are they likely to avoid such methods in their out-of-school life? It might also be noted that no one has shown that an increased facility with type-*d* problems—real problems—transfers out of school either, despite their apparent face validity.

I raise these issues about the kinds of problems to be used in instruction because, first, over the past 25 years they seem not to have been given much attention by researchers, and second, different assumptions about the instructional uses of problems seem to have led researchers in somewhat different directions. In particular, some researchers—many but by no means all of whom are psychologists—have accepted the type-*c* word problem as the vehicle for problem solving in the mathematics curriculum and have aimed their efforts at improving instruction so that more students will learn how to solve more such problems correctly. Such efforts are not to be disparaged. Many facets of the