LINEAR ALGEBRA WITH APPLICATIONS

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To
Judith

Preface

This textbook is suitable for either a sophomore level course or for a junior- or senior-level course. The only prerequisite is calculus. If the text is to be taught at the sophomore level, one should probably spend more time on the earlier chapters and omit most of the more difficult optional sections. On the other hand, if the course is to be taught on the junior or senior level, the instructor should probably spend less time on the early chapters and cover more of the optional material. The explanations in the text are given in sufficient detail so that students at either level will have little trouble reading and understanding. To further aid the student, a large number of examples have been worked out completely. Applications have been scattered throughout the text rather than put together in a chapter at the end. In this way they can be used to motivate new material and to illustrate the relevance of the material that has just been presented. When applications are included at the end of a text, they are more likely to be omitted because of lack of time.

The text tries to give fairly complete coverage to a very broad subject. Consequently, there is probably more material included than can possibly be covered in a one-quarter or one-semester course. The instructor then has some freedom in the choice of topics, and consequently the instructor may design the course to meet the needs of the class. Some instructors may decide to emphasize the mathematical theory in Chapters 3 and 4 and others may decide to skip over these chapters and spend more time on the applied topics in Chapters 5 and 6. As a general rule, if you ask n mathematicians what should go into a course, you will get n different answers. Even if many of the topics in the text are omitted, the students should get a feeling for the overall scope of the subject matter. Furthermore, many of the students may use their text later as a reference and consequently may end up learning many of the optional topics on their own; optional material is preceded by an asterisk. The following is a guide to the various chapters in the text.

Chapter 1. The first two sections deal with systems of equations. Sections 3 and 4 are concerned with matrices and matrix algebra. Most of Section 5 is optional. The instructor should cover the material at the beginning of this section, which includes an explanation of the notation used for column vectors. The material on block multiplication may be omitted at the discretion of the instructor. Block multiplication is used later in the text in some of the optional sections in Chapters 6 and 7, but it has been avoided in those sections that form the core of the text.

Chapter 2. This is a short chapter on determinants. Determinants will be used later in the text for introducing such topics as linear independence and eigenvalues.

Chapter 3. The basic theory of vector spaces is presented in this chapter. All five sections should be covered.

Chapter 4. Chapter 4 is devoted to linear transformations. Instructors wishing to present a more applied course may omit all or part of this chapter.

Chapter 5. This is a long chapter on orthogonality and its applications. Most of Sections 1, 2, 3, 6, and 7 should be covered if at all possible. Section 4 on matrix norms is optional, but it is a prerequisite for Section 7 of Chapter 6 and the last five sections of Chapter 7. Section 5, on least squares problems, is also listed as optional. This is one of the most important applications of linear algebra and is well worth covering if time permits. The last section is an optional section on orthogonal polynomials. Orthogonal polynomials play an important role in many areas of mathematics, but they never seem to get the attention they deserve in the standard courses.

Chapter 6. This chapter treats one of the most important subjects in linear algebra, eigenvalues. Sections 1 and 3 should definitely be covered. Section 2 presents one of the main applications of eigenvalues. If the instructor does not wish to cover the entire section, we recommend that the material through Example 1 be presented. Section 4 deals with matrices with complex entries. This section is optional, but it is recommended that the material in the beginning of the section be covered so that the student gets some exposure to the complex case. In this chapter the really impressive nature of the applications of linear algebra should become apparent. Sections 5 and 6 are optional sections which present some of these applications. A discussion of the Perron–Frobenius theory of nonnegative matrices is given in Section 7. The proofs are omitted. These are powerful theorems and the proofs would be way beyond the scope of a first course.

However, the theory is necessary for an understanding of the Leontief input–output models. This is one of the nicest applications of linear algebra.

Chapter 7. Although this chapter is optional, it may well be the most important chapter for students who are going to work in industry. Instructors who want to incorporate numerics into the course may consider teaching the first three sections of this chapter immediately after completing Chapter 1. Section 4 could then be taught with the section on matrix norms in Chapter 5. The section on the singular value decomposition is highly recommended. Only recently has this subject been given the recognition it deserves. This section is included in Chapter 7 because of its importance to numerical linear algebra, but it could just as well be covered in Chapter 6. The most important algorithms presented in the last two sections are more advanced in nature, and consequently they are only outlined rather than presented in detail. Part of Section 9 is theoretical and could be covered earlier in the text along with Section 7.

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S. L.

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1

Matrices and Systems of Equations

INTRODUCTION

Probably the most important problem in mathematics is that of solving a system of linear equations. It would not be conservative to estimate that well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage. Using the methods of modern mathematics, it is often possible to take a sophisticated problem and reduce it to a single system of linear equations. Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics. It seems appropriate then that this text should begin with a section on linear systems.

1. SYSTEMS OF LINEAR EQUATIONS

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables. A linear system of m equations in n unknowns is then a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where a_{ij}'s and the b_i's are all real numbers. We will refer to systems of the form (1) as $m \times n$ linear systems. The following are examples of linear systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$ (b) $\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$ (c) $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \quad &= 4 \end{aligned}$

System (a) is a 2×2 system, (b) is a 2×3 system, and (c) is a 3×2 system.

By a solution to an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations of the system. For example, the ordered pair (1, 2) is a solution to system (a), since

$$\begin{aligned} 1 \cdot (1) + 2 \cdot (2) &= 5 \\ 2 \cdot (1) + 3 \cdot (2) &= 8 \end{aligned}$$

The ordered triple (2, 0, 0) is a solution to system (b), since

$$\begin{aligned} 1 \cdot (2) - 1 \cdot (0) + 1 \cdot (0) &= 2 \\ 2 \cdot (2) + 1 \cdot (0) - 1 \cdot (0) &= 4 \end{aligned}$$

Actually, system (b) has many solutions. If α is any real number, it is easily seen that the ordered triple $(2, \alpha, \alpha)$ is a solution. However, system (c) has no solution. It follows from the third equation that the first coordinate of any solution would have to be 4. Using $x_1 = 4$ in the first two equations, we see that the second coordinate must satisfy

$$\begin{aligned} 4 + x_2 &= 2 \\ 4 - x_2 &= 1 \end{aligned}$$

Since there are no real numbers that satisfy both of these equations, the system has no solution. If a linear system has no solution, we say that the system is *inconsistent*. Thus system (c) is inconsistent, while systems (a) and (b) are both consistent.

The set of all solutions to a linear system is called the *solution set* of the system. If a system is inconsistent, its solution set is empty. A consistent system will have a nonempty solution set. To solve a consistent system, one must find its solution set.

2 × 2 Systems

Let us examine geometrically a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Each equation can be represented graphically as a line in the plane. The ordered pair (x_1, x_2) will be a solution to the system if and only if it lies on both lines. For example, consider the three systems

$$\text{(i)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \text{(ii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned} \qquad \text{(iii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

The two lines in system (i) intersect at the point (2, 0). Thus $\{(2, 0)\}$ is the solution set to (i). In system (ii) the two lines are parallel. Therefore, system (ii) is inconsistent and hence the solution set is $\varnothing$. The two equations in system (iii) both represent the same line. Any point on that line will be a solution to the system (see Figure 1.1.1).

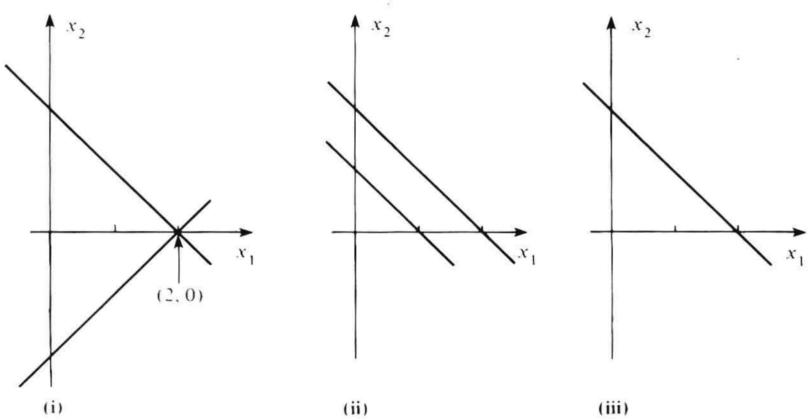

Figure 1.1.1

In general, there are three possibilities: the lines intersect at a point, they are parallel, or both equations represent the same line. The solution set then contains either one, zero, or infinitely many points.

The situation is similar for $m \times n$ systems. If a consistent system has exactly one solution, it is said to be *independent;* otherwise, it is *dependent.* We will see in the next section that if a linear system has more than one solution, it must have infinitely many solutions. The following table summarizes the three possibilities.

Type of System	*Number of Solutions*
Inconsistent	0
Consistent and independent	1
Consistent and dependent	Infinitely many

Equivalent Systems

Consider the two systems

$$\text{(a)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System (a) is easy to solve because it is clear from the last two equations that $x_2 = 3$ and $x_3 = 2$. Using these values in the first equation, we get

$$\begin{aligned} 3x_1 + 2 \cdot 3 - 2 &= -2 \\ x_1 &= -2 \end{aligned}$$

Thus the solution to the system is $(-2, 3, 2)$. System (b) seems to be more difficult to solve. Actually, system (b) has the same solution as system (a). To see this, add the first two equations of the system

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ \hline x_2 = 3 \end{array}$$

If (x_1, x_2, x_3) is any solution to (b), it must satisfy all the equations of the system. Thus it must satisfy any new equation formed by adding two of its equations. Therefore, x_2 must equal 3. Similarly, (x_1, x_2, x_3) must satisfy the new equation formed by subtracting the first equation from the third:

$$\begin{array}{r} 3x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 2x_2 - x_3 = -2 \\ \hline 2x_3 = 4 \end{array}$$

Therefore, any solution to system (b) must also be a solution to system (a). By a similar argument, it can be shown that any solution to (a) is also a solution to (b). This can be done by subtracting the first equation from the second:

$$\begin{array}{r} x_2 \qquad = 3 \\ 3x_1 + 2x_2 - x_3 = -2 \\ \hline -3x_1 - x_2 + x_3 = 5 \end{array}$$

and by adding the first and third equations:

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ 2x_3 = 4 \\ \hline 3x_1 + 2x_2 + x_3 = 2 \end{array}$$

Thus (x_1, x_2, x_3) is a solution to system (b) if and only if it is a solution to system (a). Therefore, both systems have the same solution set, $\{(-2, 3, 2)\}$.

Definition. Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Clearly, if we interchange the order in which two equations of a system are written, this will have no effect on the solution set. The reordered system will be equivalent to the original system. For example, the systems

$$\begin{array}{rl} x_1 + 2x_2 = 4 & \\ 3x_1 - x_2 = 2 & \\ 4x_1 + x_2 = 6 & \end{array} \quad \text{and} \quad \begin{array}{rl} 4x_1 + x_2 = 6 \\ 3x_1 - x_2 = 2 \\ x_1 + 2x_2 = 4 \end{array}$$

clearly have the same solution set.

If one of the equations of a system is multiplied through by a nonzero real number, this will have no effect on the solution set and the new system will be equivalent to the original system. For example, the systems

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned}$$

are equivalent.

If a multiple of one equation is added to another equation, the new system will be equivalent to the original system. This follows since the n-tuple $(x_1, \ldots, x_n)$ will satisfy the two equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

if and only if it satisfies the equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ (a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i \end{aligned}$$

To summarize, there are three operations that can be used on a system to obtain an equivalent system:

I. The order in which any two equations are written may be interchanged.
II. Both sides of an equation may be multiplied by the same nonzero real number.
III. A multiple of one equation may be added to another.

Given a system of equations, one can use these operations to obtain an equivalent system that is easier to solve.

$n \times n$ Systems

Let us restrict ourselves to $n \times n$ systems for the remainder of this section. We will show that if an $n \times n$ system is independent, then operations I and III can be used to obtain an equivalent "triangular system."

Definition. A system is said to be in **triangular form** if in the kth equation the coefficients of the first $k - 1$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, \ldots, n)$.

EXAMPLE 1. The system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

is in triangular form, since in the second equation the coefficients are 0, 1, -1, respectively, and in the third equation the coefficients are 0, 0, 2, respectively. Because of the triangular form, this system is easy to solve. It follows from the third equation that $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2 \qquad \text{or} \qquad x_2 = 4$$

Using $x_2 = 4$, $x_3 = 2$ in the first equation, we end up with

$$\begin{aligned} 3x_1 + 2 \cdot 4 + 2 &= 1 \\ x_1 &= -3 \end{aligned}$$

Thus the solution to the system is $(-3, 4, 2)$.

Any $n \times n$ triangular system can be solved in the same manner as the last example. First, the nth equation is solved for the value of x_n. This value is used in the $(n-1)$st equation to solve for x_{n-1}. The values x_n and x_{n-1} are used in the $(n-2)$nd equation to solve for x_{n-2}, and so on. We will refer to this method of solving a triangular system as *back substitution.*

EXAMPLE 2. Solve the system

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ x_2 - 2x_3 + 3x_4 &= 2 \\ 4x_3 + 3x_4 &= 3 \\ 4x_4 &= 4 \end{aligned}$$

SOLUTION. Using back substitution, we obtain

$$\begin{aligned} 4x_4 &= 4 & x_4 &= 1 \\ 4x_3 + 3 \cdot 1 &= 3 & x_3 &= 0 \\ x_2 - 2 \cdot 0 + 3 \cdot 1 &= 2 & x_2 &= -1 \\ 2x_1 - (-1) + 3 \cdot 0 - 2 \cdot 1 &= 1 & x_1 &= 1 \end{aligned}$$

Thus the solution is $(1, -1, 0, 1)$.

If a system of equations is not triangular, we will use operations I and III to try to obtain an equivalent system that is in triangular form.