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Mathematics of Physics and Modern Engineering

MATHEMATICS OF PHYSICS AND MODERN ENGINEERING

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PREFACE

The rapidly decreasing time lag between scientific discoveries and applications imposes ever-increasing demands on the mathematical equipment of scientists and engineers. Although the mathematical preparation of engineering students has been strengthened materially in the past thirty years, the introduction of courses beyond the traditional "terminal course" in calculus has been largely confined to a few leading institutions. The reluctance to broaden significantly the program of instruction in mathematics can be attributed in part to the crowded engineering curricula, in part to the failure to sense the central position of mathematics in sciences and technology, and in part to the scarcity of suitable staffs and instructional media. The broadening, however, is inevitable, for it is now generally recognized that no professional engineer can keep abreast of scientific developments without substantially extending his mathematical horizons.

This book, in common with its predecessor written by the senior author some twenty-five years ago, has as its main aim a sound extension of such horizons. The authors not only have been guided by their subjective appraisal of the live present-day needs of the engineering profession but have also taken into account the views of the leaders of engineering thought as expressed in numerous conferences and symposia on engineering education sponsored by the National Science Foundation, the American Society of Engineering Education, and its predecessor the Society for the Promotion of Engineering Education.

There are many conflicting and often prejudiced currents of thought as to how mathematics should be presented to students of applied sciences. Some believe that mathematics is one whole and indivisible and hence should be presented unto all alike, regardless of the differing creeds. Others are content with a catalogue of useful formulas, rules, and devices for solving problems. The authors think that these two extreme viewpoints are somewhat limited, since they recognize only two of the many facets of mathematics. A preoccupation with the logic of mathematics and the overemphasis of a convention called rigor are among the best known means for stifling interest in mathematics as a crutch to common sense. On the other hand, a presentation which puts applications above the medium making applications possible is sterile, because it gives no inkling of the supreme importance of generalizations and abstractions in applications. The authors have tried to strike a balance which would make this book both a sound and an inspiring introduction to applied mathematics.

The material in this book appears in nine chapters, each of which is complete and virtually independent of the others. Occasional cross references to other chapters are intended to correlate the topics and to enhance the usefulness of the book as a reference volume. Each chapter is subdivided into functional parts, many of which also form an organized whole. The earlier parts of each chapter are less advanced and should serve as an introduction to more difficult topics treated in the later parts. The text material set in small type usually deals with generalizations and develops the less familiar concepts which are sure to grow in importance in applications.

The choice of topics is based on the authors' estimate of the frequency with which the subjects treated occur in applications. The illustrative material, examples, and problems have been chosen more for their value in emphasizing the underlying principles than as a collection of instances of dramatic uses of mathematics in specific situations confronting practicing engineers.

Although the book is written so as to require little, if any, outside help, the reader is cautioned that no amount of exposition can serve as a substitute for concentration in following the course of the argument in a serious discipline. In order to facilitate the understanding of the principles and to cultivate the art of formulating physical problems in the language of mathematics, numerous illustrative examples are worked out in detail. The authors believe with Newton that *exempla non minus docecunt quam precaepta.*

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TO THE INSTRUCTOR

In the sense that a working course in calculus is the sole technical prerequisite, this book is suitable for the beginner in applied mathematics. But when viewed in the light of the present-day requirements of the engineering profession, the text includes a large amount of material of direct interest to practicing engineers.

It is certain that within the next twenty years the methods of functional analysis and, in particular, the Hilbert space theory will be in general use in technology. A foundation for the assimilation of the function-space concepts should be laid now, and we did not hesitate to do so in several places in this book.

We have arranged the contents in nine independent chapters which, in turn, are subdivided into parts, most of which can be read independently of the rest. The earlier parts of each chapter are less advanced, and our experience has shown that several introductory courses for students of science and technology can be based on the material contained in the earlier parts. When taken in sequence, this book has ample substance for four consecutive semester courses meeting three hours a week.

This book is also suitable for courses in mathematical analysis bearing such labels as ordinary differential equations, partial differential equations, vector analysis, advanced calculus, complex variable, and so on.

Thus Chap. 1, when supplemented by Secs. 12 to 14 of Chap. 2, has adequate material for a solid semester course in ordinary differential equations. Instructors wishing to include an introduction to numerical methods of solutions of differential equations will find suitable material in Secs. 14 to 18 of Chap. 9. The use of Laplace transforms in solving differential equations is discussed in Appendix B, which includes, among other things, a meaningful introductory presentation of the "Dirac delta function."

Chapter 6, together with Secs. 18 to 25 of Chap. 2, has ample material for a semester course in partial differential equations.

Chapters 4 and 5 have sufficient content for a modern course in vector analysis.

Chapter 7, preceded by the relevant topics on line integrals in Chap. 5, is adequate for an introductory course in complex variable theory.

Chapter 8 can be used in a semester course on probability theory and applications meeting two hours a week. A course entitled "Probability and Numerical Methods" meeting three hours a week can be based on the material in Chaps. 8 and 9.

Although this book was written primarily for students of physical sciences, it is unlikely that a liberal arts student who followed it in an advanced calculus course would be obliged to "unlearn" anything in his subsequent studies.

The contents of this book include what we believe should be the minimum mathematical equipment of a scientific engineer. It may not be out of place to note that the mathematical preparation of physicists and engineers in Russia exceeds the minimum laid down here. While the curricula of only a few leading American engineering colleges provide now for more than one year of mathematics beyond calculus, their number will continue to increase with the realization that the time allotted to mathematics is a sound capital investment, yielding excellent returns both in the time gained in professional studies and in the depth of penetration.

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CHAPTER 1

ORDINARY DIFFERENTIAL EQUATIONS

Preliminary Remarks and Orientation

The Solution of First-order Equations

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Systems of Equations

The power and effectiveness of mathematical methods in the study of natural sciences stem, to a large extent, from the unambiguous language of mathematics, with the aid of which the laws governing natural phenomena can be formulated. Many natural laws, especially those concerned with rates of change, can be phrased as equations involving derivatives or differentials. For example, when a verbal statement of Newton's second law of motion is translated into mathematical symbols, there results an equation relating time derivatives of displacements to forces. A study of such equations then provides a complete qualitative and quantitative characterization of the behavior of mechanical systems under the action of forces. Several broad types of equations studied in this book characterize physical situations of great diversity and practical interest.

The first half of this chapter is concerned with preliminaries and special techniques devised for the solution of the first-order equations arising commonly in applications. The second half contains a comprehensive treatment of linear differential equations with constant coefficients and an introduction to linear equations with variable coefficients. Linear equations occupy a prominent place in the study of the response of elastic structures to impressed forces and in the analysis of electrical circuits and servomechanisms. They also appear in numerous boundary-value problems in the theory of diffusion and heat flow, in quantum mechanics and fluid mechanics, and in electromagnetic theory.

PRELIMINARY REMARKS AND ORIENTATION

1. Definition of Terms and Generalities. Any function containing variables and their derivatives (or differentials) is called a *differential expression,* and every equation involving differential expressions is called a *differential equation.* Differential equations are divided into two classes, *ordinary* and *partial.* The former contain only one independent variable and derivatives with respect to it. The latter contain more than one independent variable.

The order of the highest derivative contained in a differential equation is called the *order* of the differential equation. Thus

$$\left(\frac{d^2y}{dx^2}\right)^4 + 3\frac{dy}{dx} + 5y^2 = 0$$

is an ordinary differential equation of order 2, and

$$\left(\frac{\partial^3 y}{\partial t^3}\right)^2 + 3\frac{\partial^2 y}{\partial x\, \partial t} + yxt = 0$$

is a partial differential equation of order 3.

A function $y = \varphi(x)$ is said to be a *solution* of the differential equation

$$F(x,y,y') = 0, \tag{1-1}$$

if, on the substitution of $y = \varphi(x)$ and $y' = \varphi'(x)$ in the left-hand member of (1-1), the latter vanishes identically.[1] Again, $y = \varphi(x)$ is a solution of the second-order equation $F(x,y,y',y'') = 0$ when the substitution $y = \varphi(x)$, $y' = \varphi'(x)$, $y'' = \varphi''(x)$ reduces this to an identity in x. Similarly for equations of order n.

For example, the first-order differential equation

$$y' + 2xy - e^{-x^2} = 0 \tag{1-2}$$

has a solution $y = xe^{-x^2}$, because the substitution of $y = xe^{-x^2}$ and $y' = e^{-x^2} - 2x^2e^{-x^2}$ in (1-2) reduces it to an identity $0 \equiv 0$. Also, the equation

$$y'' + y = 0$$

has a solution $y = \sin x$, as can be easily verified by substitution.

We begin our study of differential equations with the first-order equation (1-1), which we suppose can be solved for y' to yield the equation

$$y' = f(x,y). \tag{1-3}$$

For reasons which will become clear presently, we shall always assume that $f(x,y)$ is a continuous function throughout some region in the xy plane, and we shall study the solutions of (1-3) [or, equivalently, of (1-1)] in that region.

The geometrical meaning of the term *solution* of (1-3) is suggested at once by the interpretation of the derivative y' as the slope of the tangent line to some curve $y = \varphi(x)$, for if (x,y) is a point on the curve $y = \varphi(x)$,

[1] Here, as elsewhere in this book, primes are used to denote differentiation: $y' \equiv dy/dx$, $y'' \equiv d^2y/dx^2, \ldots, y^{(n)} \equiv d^ny/dx^n$.

and if at every point of this curve the slope is equal to $f(x,y)$, then $\varphi(x)$ is a solution of (1-3).

One can get an idea of the shape of the curve $y = \varphi(x)$ in the following way: Let us choose a point (x_0,y_0) and compute

$$y' = f(x_0,y_0). \tag{1-4}$$

The number $f(x_0,y_0)$ determines a direction of the curve at (x_0,y_0). Now, let (x_1,y_1) be a point near (x_0,y_0) in the direction specified by (1-4). Then $y' = f(x_1,y_1)$ determines a new direction at (x_1,y_1) (Fig. 1). Upon proceeding a short distance in this new direction, we select a new point (x_2,y_2) and at this point determine a new slope $y' = f(x_2,y_2)$. As this process is continued, a curve is built up consisting of short line segments.

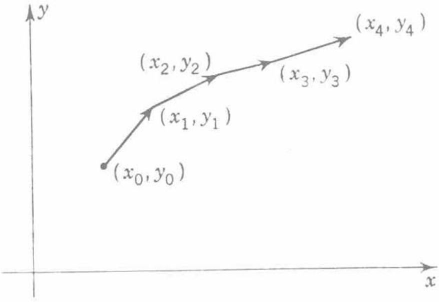

FIG. 1

If the successive points (x_0,y_0), (x_1,y_1), (x_2,y_2), . . . , (x_n,y_n) are chosen near one another, the series of straight-line segments approximates a smooth curve $y = \varphi(x)$ which is a solution of (1-3) associated with the choice of the initial point (x_0,y_0). A different choice of the initial point will, in general, give a different curve, so that the solutions of Eq. (1-3) can be viewed as being given by a whole family of curves. Such curves are called *integral curves*, and each curve in the family represents a *particular solution* or an *integral* of our equation.

Also, we can make a surmise that, unless $f(x,y)$ in the right-hand member of (1-3) is a badly behaving function, for each choice of the initial point there will be just one solution of Eq. (1-3). This surmise is capable of proof, which we do not give here because it requires the use of analytical tools which are not provided in the usual calculus courses. However, the statement of essential facts is easy to grasp, and since it will facilitate the understanding of subsequent developments, we give it here as a basic theorem.

EXISTENCE AND UNIQUENESS THEOREM. *The equation $y' = f(x,y)$ has one and only one integral curve passing through each point of the region in which both $f(x,y)$ and $\partial f/\partial y$ are continuous functions.*[1]

Unless a statement to the contrary is made, we shall suppose that the restrictions imposed on $f(x,y)$ in this theorem are fulfilled, so that Eq.

[1] It suffices to suppose that $|\partial f/\partial y|$ is bounded in the region. Proofs of this theorem are contained in many books on differential equations, for example, E. L. Ince, "Ordinary Differential Equations," p. 62. See also Sec. 17 of this chapter.

(1-3) has a unique solution for each choice of (x_0,y_0) in the appropriate region of the xy plane.

Since by changing the initial value $y|_{x=x_0} = y(x_0)$ we get a family of curves depending on the arbitrarily chosen value $y(x_0)$, the equation of this family can be written in the form

$$y = \varphi(x,c) \tag{1-5}$$

involving one arbitrary constant c, corresponding to the arbitrary choices of $y(x_0)$. A particular curve of the family (1-5) passing through (x_0,y_0) is then determined by the value of c such that $y_0 = \varphi(x_0,c)$.

A solution of the first-order equation (1-3) involving one arbitrary constant is called a *general solution*.[1] Such solutions are often written in the implicit form

$$\Phi(x,y,c) = 0, \tag{1-6}$$

where it is understood that (1-6) can be solved for y to yield the explicit form (1-5). In practice it may not be necessary to exhibit the explicit form. The essential feature of the *general solution* [be it given by (1-5) or (1-6)] is that the constant c in it can be determined so that an integral curve passes through a given point (x_0,y_0) of the region under consideration.

We illustrate this by demonstrating that throughout the xy plane the general solution of Eq. (1-2) can be written as

$$y = e^{-x^2}(x + c). \tag{1-7}$$

The fact that (1-7) is, indeed, a solution is easily verified by substituting (1-7) in (1-2). Moreover, it is a general solution, because on setting $x = x_0$ and $y = y_0$ we get

$$y_0 = e^{-x_0^2}(x_0 + c). \tag{1-8}$$

Thus the integral curve passing through (x_0,y_0) corresponds to

$$c = y_0e^{x_0^2} - x_0.$$

As another example consider the equation

$$\frac{dy}{dx} = f(x), \tag{1-9}$$

where $f(x)$ is any continuous function. A *general* solution of this equation, obtained by direct integration, is

$$y = \int f(x)\,dx + c. \tag{1-10}$$

[1] Some first-order equations may have solutions which cannot be determined from the general solution for any value of c. Such solutions, called *singular solutions*, arise only when the conditions imposed on $f(x,y)$ in the basic theorem are not fulfilled.

We show next that (1-10) is a general solution of (1-9). We denote an indefinite integral in (1-10) by $F(x)$, so that $dF/dx = f(x)$. Then (1-10) is the same as

$$y = F(x) + c. \tag{1-11}$$

On setting $x = x_0$, $y = y_0$, we get

$$y_0 = F(x_0) + c,$$

so that

$$c = y_0 - F(x_0),$$

and we can, therefore, write (1-11) as

$$y = F(x) - F(x_0) + y_0$$

$$\equiv F(x)\Big|_{x_0}^{x} + y_0. \tag{1-12}$$

But from the fundamental theorem of integral calculus,

$$\int_{x_0}^{x} f(x)\,dx = F(x)\Big|_{x_0}^{x}$$

and therefore (1-12) yields the desired particular solution

$$y = \int_{x_0}^{x} f(x)\,dx + y_0, \tag{1-13}$$

corresponding to the choice of the initial point (x_0,y_0).

Formula (1-13) illustrates the procedure of deducing particular solutions by integrating the given equation (1-9) between limits. It is frequently simpler than the procedure of determining the desired solution by calculating the constant c in the general solution from the initial data.

The foregoing discussion can be extended to equations of higher order. Thus, *the nth-order equation*

$$F(x,y,y',\ldots,y^{(n)}) = 0, \tag{1-14}$$

which we shall write in the form solved for $y^{(n)}$ as

$$y^{(n)} = f(x,y,y',\ldots,y^{(n-1)}), \tag{1-15}$$

has a unique solution for n arbitrarily assigned initial values,

$$y(x_0),\ y'(x_0),\ldots,\ y^{(n-1)}(x_0), \tag{1-16}$$

whenever the function f in (1-15) *is continuous together with the partial derivatives* $\partial f/\partial y$, $\partial f/\partial y'$, ..., $\partial f/\partial y^{(n-1)}$.

When the values in (1-16) are varied, we get a family of curves, the so-called *n-parameter family*, corresponding to n independent choices of constants in (1-16). The equation of this family of solutions can be written in the form

$$y = \varphi(x,c_1,c_2,\ldots,c_n) \tag{1-17}$$