PROCEEDINGS OF
SYMPOSIA IN PURE MATHEMATICS

VOLUME II

LATTICE THEORY

AMERICAN MATHEMATICAL SOCIETY

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PREFACE

This volume contains the papers presented at the Symposium on Partially Ordered Sets and Lattice Theory held in conjunction with the Monterey meeting of the American Mathematical Society in April 1959. The Symposium was sponsored by the American Mathematical Society and supported by a grant from the National Science Foundation. The interest and support of these organizations is gratefully acknowledged.

Some twenty-one years earlier, on April 15, 1938, the first general symposium on lattice theory was held in Charlottesville in conjunction with a regular meeting of the American Mathematical Society. The three principal addresses on that occasion were entitled: Lattices and their Applications, On the Application of Structure Theory to Groups, and The Representation of Boolean Algebras. It is interesting to observe that the first and last of these titles appear again as section titles for the present Symposium. Furthermore the second title is still of current interest as evidenced by the paper of Marshall Hall. Nevertheless there have been major changes in emphasis and interest during the intervening years and thus some general comments concerning the present state of the subject and its relationship to other areas of mathematics appear to be appropriate.

The theory of groups provided much of the motivation and many of the technical ideas in the early development of lattice theory. Indeed it was the hope of many of the early researchers that lattice-theoretic methods would lead to the solution of some of the important problems in group theory. Two decades later, it seems to be a fair judgment that, while this hope has not been realized, lattice theory has provided a useful framework for the formulation of certain topics in the theory of groups (for example, generalizations of the Jordan-Hölder theorem) and has produced some interesting and difficult group-theoretic problems (cf. the excellent monograph of M. Suzuki). On the other hand, the fundamental problems of lattice theory have, for the most part, not come from this source but have arisen from attempts to answer intrinsically natural questions concerning lattices and partially ordered sets; namely, questions concerning the decompositions, representations, imbedding, and free structure, of such systems. It should be pointed out that group theory and other areas of mathematics have furnished concepts and methods which have proved to be useful in the study of these questions. Thus the techniques associated with the study of composition series and chief series in group have been successfully applied to the structure of modular and semi-modular lattices. Set topology and ring theory have been the source of many fruitful ideas in the study of Boolean algebras. Also the theory of linear vector spaces and projective geometries have contributed some of the basic methods for the development of the theory of complemented modular lattices and, in particular, continuous geometries. Nevertheless, as the study of these basic questions has progressed, there has come into being a sizable body of technical ideas and methods which are peculiarly lattice-theoretic in nature. These conceptual tools are intimately related to the underlying order relation and are particularly appropriate for the study of general lattice structure.

At the 1938 Symposium, lattice theory was described as a "vigorous and promising younger brother of group theory". In the intervening years it has developed into a full-fledged member of the algebraic family with an extensive body of knowledge and a collection of exciting problems all of its own. Such outstanding problems as the construction of a set of structure invariants for certain classes of Boolean algebras, the characterization of the lattice of congruence relations of a lattice, the imbedding of finite lattices in finite partitions lattices, the word problem for free modular lattices, and the construction of a dimension theory for continuous, non-complemented, modular lattices, have an intrinsic interest independent of the problems associated with other algebraic systems. Furthermore, these and other current problems are sufficiently difficult that imaginative and ingenious methods will be required in their solutions. A vigorous group of mathematicians are attacking these problems and the results of some recent progress may be found in the papers included in this volume.

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PART I

LATTICE STRUCTURE THEORY

STRUCTURE AND DECOMPOSITION THEORY OF LATTICES

BY

R. P. DILWORTH

1. Introduction. One of the most natural problems which arise in the investigation of an abstract algebraic system is that of representing the elements of the system in terms of a canonical subset by means of the operations of the system. Thus for a polynomial domain over a field with the operation that of ordinary polynomial multiplication it is the problem of representing polynomials as products of irreducible polynomials. For lattices there are two operations with respect to which we may consider the representation of the elements of the lattice. Since the operations are dual it suffices to consider representations with respect to one of the operations. Thus we shall treat only meet representations. Now an element which cannot be expressed as the meet of elements distinct from itself clearly has only trivial representations. Furthermore these elements must surely be included in any reasonable canonical set. Thus we shall be particularly concerned with meet representations in terms of meet irreducible elements.

A second natural problem which arises in any algebraic investigation is that of representing the system as a whole in terms of certain distinguished subsystems by means of canonical constructions. The most familiar of these constructions is the direct union (direct product) representation. Again, if possible, it is desirable to represent the system as a direct or subdirect union of systems which cannot be further decomposed, i.e., indecomposable systems.

For most algebraic systems, these two problems are quite different in character and the results in one case may have very little connection with the results in the other. For lattices, on the other hand, these two problems are intimately related. For direct and subdirect union representations of lattices, or indeed of any algebraic system, can be described in terms of the structure of the congruence relations on the algebraic system. But the congruence relations in a very natural way form a lattice. Furthermore, the meet representations of the null congruence relation correspond to the subdirect representations of the algebraic system. Hence decomposition theorems for the elements of a class of lattices will immediately give representation theorems for those algebraic systems having congruence lattices belonging to the given class of lattices. In particular, decomposition theorems which hold for the null element of the lattice of congruence relations of a lattice lead to subdirect (or direct) union representation theorems for the lattice itself. Structure theorems for lattices are greatly simplified by the fact that the lattice of congruence relations on a lattice is always distributive. Thus it suffices to study the decomposition theory of distributive lattices in developing the structure theory of arbitrary lattices. However, applications to the structure of other algebraic systems require a decomposition theory for lattices of a much more general type. This paper will be devoted to a description of the relationship between structure and decomposition theorems followed by an account of the development of decomposition theory up to present. The final section contains a discussion of some of the outstanding problems of current interest in this area of lattice theory.

2. **Direct and subdirect unions.** Consider a collection of lattices L_α where α belongs to an index set A. $\prod_\alpha L_\alpha$ will denote the "Cartesian product" of the lattices L_α; that is the set of functions f on A such that $f(\alpha) \in L_\alpha$ for each $\alpha \in A$. $\prod_\alpha L_\alpha$ is a lattice if we define

$$(f \cup g)(\alpha) = f(\alpha) \cup g(\alpha),$$
$$(f \cap g)(\alpha) = f(\alpha) \cap g(\alpha).$$

A lattice L is "imbedded" in $\prod_\alpha L_\alpha$ if L is isomorphic to a sublattice of $\prod_\alpha L_\alpha$. L is a *subdirect union* of the lattices L_α if L is imbedded in $\prod_\alpha L_\alpha$ and for each $a_\alpha \in L_\alpha$ there exists a function $f \in \prod_\alpha L_\alpha$ corresponding to an element of L such that $f(\alpha) = a_\alpha$. Finally L is a *direct union* of the lattices L_α if L is imbedded in $\prod_\alpha L_\alpha$ and for every finite set of indices $\alpha_1, \cdots, \alpha_n$ and elements $a_{\alpha_1}, \cdots, a_{\alpha_n}$ in $L_{\alpha_1}, \cdots, L_{\alpha_n}$ respectively, there exists f corresponding to an element of L such that $f(\alpha_i) = a_{\alpha_i}$, $i = 1, \cdots, n$.

If L is imbedded in $\prod_\alpha L_\alpha$, this imbedding induces a natural set of congruence relations on L; namely

$$a \;\theta_\alpha\; b \text{ if and only if } a_\alpha = b_\alpha.$$

For the study of lattice structure it is useful to have a characterization of imbedding, subdirect unions, and direct unions in terms of properties of this set of congruence relations. We begin by pointing out that the set $\Theta(L)$ of congruence relations on L can be partially ordered by the relation

$$\theta \leqq \phi \text{ if and only if } a \;\theta\; b \Rightarrow a \;\phi\; b.$$

$\Theta(L)$ has a unique maximal element ι, namely the congruence relation which identifies all elements of L. $\Theta(L)$ likewise has a unique minimal element ω, namely the equality relation on L. Under the above partial ordering $\Theta(L)$ is a complete lattice. If $\Phi \subseteq \Theta(L)$ then the meet and join of the subset Φ may be characterized as follows:

$a \bigcap \Phi\, b$ if and only if $a \;\phi\; b$ for all $\phi \in \Phi$;

$a \bigcup \Phi\, b$ if and only if $a = a_0, a_1, \cdots, a_m = b$ exist such that $a_{i-1} \;\phi_i\; a_i$ for $\phi_i \in \Phi$.

In addition to the lattice operations, a permutability relation plays an important role in structure theorems. Congruence relations θ and ϕ are said to *permute* if $a \;\theta\; b$ and $b \;\phi\; c$ imply the existence of d such that $a \;\phi\; d$ and $d \;\theta\; c$.

Permutability is preserved by the lattice operations. If θ permutes with both ϕ and ψ, then θ permutes with $\phi \cup \psi$ and $\phi \cap \psi$. Furthermore if θ permutes with all of the congruence relations belonging to Φ, then θ permutes with $\bigcup \Phi$. It is not difficult to show that any two congruence relations on a relatively complemented lattice permute. Another useful result asserts that any lattice of permuting congruence relations on an arbitrary algebraic system is modular. For lattices there is the stronger theorem of Funayama [**9**] that $\Theta(L)$ is always distributive.

Now if θ is a congruence relation on L, the congruence classes form a lattice which is a homomorphic image of L. The lattice of congruence classes will be denoted by $\theta(L)$. If L is imbedded in $\prod_\alpha L_\alpha$ and θ_α is the congruence relation determined by the component L_α, then each congruence class of θ_α determines a unique $a_\alpha \in L_\alpha$ and $\theta_\alpha(L)$ is thus a sublattice of L_α. Furthermore if $a\ \theta_\alpha\ b$ for all $\alpha \in A$, then $a_\alpha = b_\alpha$ all $\alpha \in A$ and hence $a = b$. Thus $\bigcap_\alpha \theta_\alpha = \omega$. If L is a subdirect union of the lattices L_α, then $\theta_\alpha(L) = L_\alpha$. Thus subdirect union representations are characterized by meet decompositions of the minimal congruence relation. Clearly subdirect union representations in terms of subdirectly irreducible lattices correspond to decompositions of ω into indecomposable congruence relations.

We next observe that if L is a direct union of the lattices L_α and α, β are two distinct elements of A, then for any pair of elements $a, b \in L$, there exists $c \in L$ such that $c_\alpha = a_\alpha$ and $c_\beta = b_\beta$. Thus $a\ \theta_\alpha\ c$ and $c\ \theta_\beta\ b$. It follows that θ_α and θ_β permute and $\theta_\alpha \cup \theta_\beta = \iota$. Hence direct union representations of L correspond to decompositions $\omega = \bigcap_\alpha \theta_\alpha$ where $\{\theta_\alpha | \alpha \in A\}$ is a set of permuting, coprime congruence relations. Conversely, any such decomposition of ω leads to a direct union representation of L.

Finally, it should be mentioned that Hashimoto [**10**] has characterized Cartesian product representations in terms of properties of $\Theta(L)$ and a generalized notion of permutability. If $\{\theta_\alpha | \alpha \in A\}$ is a set of congruence relations, let $\theta_\alpha^* = \bigcap \{\theta_\beta | \beta \neq \alpha\}$. The set $\{\theta_\alpha | \alpha \in A\}$ is said to be *completely coprime* if $\theta_\alpha \cup \theta_\alpha^* = \iota$ for all α. Likewise the set $\{\theta_\alpha | \alpha \in A\}$ is said to be completely permutable if $a_\alpha(\theta_\beta^* \cup \theta_\beta^*)a_\beta$ for all α and β, implies that there exists $a \in L$ such that $a\theta_\alpha a_\alpha$ for all $\alpha \in A$. Then if $\{\theta_\alpha | \alpha \in A\}$ is a set of completely coprime and completely permutable congruence relations, L is isomorphic to $\prod_\alpha \theta_\alpha(L)$. It is also easy to see that the congruence relations on a Cartesian product determined by the component lattices form a set of completely coprime and completely permutable congruence relations.

3. The classical decomposition theorems. As mentioned in the introduction and in view of the applications to structure problems we shall be interested in the representation of elements of a lattice as meets of indecomposable elements. However, the appropriate definition of indecomposability will depend upon the type of representations under consideration. The two principal definitions are the following.

DEFINITION 3.1. q is *(meet) irreducible* if $q = x \cap y$ implies $q = x$ or $q = y$.

DEFINITION 3.2. q is *completely (meet) irreducible* if $q = \bigcap S$ implies $q = s$ for some $s \in S$.

In general, the first definition is appropriate for finite decompositions while the second is the appropriate concept for infinite decompositions.

The fundamental problems in decomposition theory concern the question of existence and uniqueness of decompositions into irreducibles. If an element of a lattice can be represented as a meet of the set Q of indecomposables and if the removal of one or more elements from the set Q gives a set with the same meet, then the representation $a = \bigcap Q$ can hardly be unique in any reasonable sense. Hence we shall restrict our attention to irredundant decompositions.

DEFINITION 3.3. A decomposition $a = \bigcap Q$ is *irredundant* if $\bigcap(Q - q) > a$ for all $q \in Q$.

If $a = q_1 \cap \cdots \cap q_n$ is a finite decomposition of a, then by deleting superfluous q's this decomposition can always be refined to an irredundant decomposition. For infinite decompositions, this process breaks down, and hence the construction of irredundant infinite decompositions requires a more elaborate procedure.

The classical existence theorem is the following.

THEOREM 3.1. *If L satisfies the ascending chain condition, then every element of L has a finite irredundant decomposition into irreducibles.*

For if the theorem is not true, the lattice L contains a maximal element a which cannot be represented as an irredundant meet of irreducibles. a is reducible since otherwise it would have an irredundant decomposition consisting of one irreducible. Hence $a = x \cap y$ where $x > a$ and $y > a$. But then both x and y may be represented as meets of irreducibles and hence a can likewise be so represented. Refining this decomposition into an irredundant one contradicts the definition of a.

The classical uniqueness theorem is due to Birkhoff [**2**] and concerns distributive lattices.

THEOREM 3.2. *If an element of a distributive lattice has a finite irredundant decomposition into irreducibles, this decomposition is unique.*

For if $a = q_1 \cap \cdots \cap q_m = q'_1 \cap \cdots \cap q'_n$ are two irredundant decompositions into irreducibles, then $q_i = q_i \cup a = q_i \cup (q'_1 \cap \cdots \cap q'_n) = (q_i \cup q'_1) \cap \cdots \cap (q_i \cup q'_n)$ and hence $q_i = q_i \cup q'_j$ for some j. Thus $q_i \geqq q'_j$ and similarly $q'_j \geqq q_k$ for some k. Since the decompositions are irredundant $i = k$ and hence $q_i = q'_j$. Similarly each q'_j is equal to q_k for some k and hence the two decompositions are identical.

Another type of uniqueness theorem is concerned with the replacement of irreducibles in one decomposition by suitably chosen irreducibles in another decomposition. The classical theorem of this type is due to Kurosch [**11**] and Ore [**12**].

THEOREM 3.3. *Let $a = q_1 \cap \cdots \cap q_m = q'_1 \cap \cdots \cap q'_n$ be two irredundant decompositions of an element of a modular lattice. Then for each q_i there exists a q'_j such that $a = q_i \cap \cdots \cap q_{i-1} \cap q'_j \cap q_{i+1} \cap \cdots \cap q_m$ is an irredundant decomposition of a.*

For if $q_i^* = q_1 \cap \cdots \cap q_{i-1} \cap q_{i+1} \cap \cdots \cap q_m$, then it can be easily verified that $[q_i \cup (q_i^* \cap x)] \cap [q_i \cup (q_i^* \cap y)] = q_i \cup (q_i^* \cap x \cap y)$. Repeated application of this formula gives $[q_i \cup (q_i^* \cap q'_1)] \cap \cdots \cap [q_i \cup (q_i^* \cap q'_n)] = q_i \cup (q_i^* \cap q'_1 \cap \cdots \cap q'_n) = q_i \cup (q_i^* \cap a) = q_i \cup a = q_i$. Since q_i is irreducible we have $q_i = q_i \cup (q_i^* \cap q'_j)$ for some j and hence $q_i \geqq q_i^* \cap q'_j$ for some j. But then $a = q_i \cap q_i^* \geqq q_i^* \cap q'_j \geqq a$ and hence $a = q_i^* \cap q'_j$. It is easy to see that this representation is irredundant.

The property expressed in this theorem we shall call the *replacement property* for irredundant decompositions in modular lattices. Repeated application of the replacement property shows that $m = n$ and hence that the number of irreducibles in the irredundant decompositions of an element of a modular lattice is unique.

We note at this point that Theorem 3.3 can be sharpened to give a simultaneous replacement theorem. Namely, the q'_j may be remembered so that each q_i can be replaced by q'_i.

The methods of proof for the classical uniqueness and replacement theorems do not extend to more general lattices. Thus a new technique is required. We will begin by discussing the finite dimensional case.

4. **Finite dimensional lattices.** Let L be a finite dimensional lattice and let $a = q_1 \cap \cdots \cap q_m$ be an irredundant decomposition into irreducibles. Then $q_i^* = q_i \cap \cdots \cap q_{i-1} \cap q_{i+1} \cap \cdots \cap q_m \neq a$ for each i and hence there exists p_i such that $q_i^* \geqq p_i \succ a$ when $p_i \succ a$ signifies that p_i covers a. Thus irredundant decompositions are closely related to properties of the elements covering a. Let u_a denote the join of all elements p covering a. Then $u_a \cap x = a$ if and only if $x = a$. Hence $a = q_1 \cap \cdots \cap q_m$ is an irredundant decomposition of a if and only if $a = (u_a \cap q_1) \cap \cdots \cap (u_a \cap q_m)$ is an irredundant decomposition of a in the quotient lattice u_a/a. Thus the study of the irredundant decompositions of a is reduced to the study of the irredundant decompositions $a = s_1 \cap \cdots \cap s_m$ in u_a/a where $s_i = q_i \cap u_a$ for some irreducible q_i of L. Now the maximal elements of u_a/a always have this form. For if $u_a \succ s$, let q be a maximal element in L such that $q \geqq s$, $q \ngeqq u_a$. Then q is irreducible and $q \cap u_a = s$ since $u_a \succ s$. In most cases of interest, the maximal elements of u_a/a are the only elements of u_a/a having this form and hence the study of the irredundant decomposition of a is

reduced to the study of the irredundant representations of a as a meet of maximal elements of u_a/a.

Now if the elements of a lattice L have unique irredundant decompositions into irreducibles, the lattice must be semimodular. For if $a \succ a \cap b$ while $a \cup b > c > b$ there exists an irreducible q_1 such that $q_1 \geqq c$, $q_1 \ngeqq a \cup b$ and an irreducible q_2 such that $q_2 \geqq b$, $q_2 \ngeqq c$. Hence $a \cap b = a \cap q_1 = a \cap q_2$. If we take any finite decomposition of a (which exists by Theorem 3.1) this leads to two decompositions of $a \cap b$ which may then be refined to irredundant decompositions. Since $q_1 \geqq c$ and $q_2 \ngeqq c$, it follows that these two irredundant decompositions are distinct contrary to assumption.

Thus in characterizing finite dimensional lattices with unique irredundant representations we may restrict our attention to semimodular lattices. But in a semimodular lattice every maximal independent set of points of u_a/a has the same number of elements, such a set of points generate a Boolean algebra, and the maximal elements of the Boolean algebra are elements covered by u_a in u_a/a. Hence there exist irredundant decompositions having the same number of components as the number of elements in a maximal independent set of points of u_a/a. Now if q is an irreducible of L such that $q \ngeqq u_a$, let $p_1, \cdots, p_k$ be a maximal independent set of points of $q \cap u_a/a$. Extend this set to a maximal set $p_i, \cdots, p_k, \cdots, p_n$ of points of u_a/a. If $s_i = p_i \cup \cdots \cup p_{i-1} \cup p_{i+1} \cup \cdots \cup p_n$, then $u_a \succ s_i$ and hence there exists q_i such that $q_i \cap u_a = s_i$. Then it is easily verified that $a = q \cap q_1 \cap \cdots \cap q_n$ is an irredundant decomposition of a. Hence if the number of components is unique it follows that $k + 1 = n$ and hence $u_a \succ q \cap u_a$. Thus the number of components in the irredundant decompositions of a will be unique only if the number of components in the irredundant representations of a as meets of elements covering u_a is unique. But it can be shown that the number of components in such decompositions is unique if and only if u_a/a is modular. Furthermore, if u_a/a is modular it follows from the classical replacement theorem (Theorem 3.3) applied to u_a/a that the replacement property holds for the irredundant decompositions of a. Finally if the irredundant decomposition of a is unique, then u_a/a consists precisely of the elements generated by a maximal independent set of points of u_a/a and hence is a Boolean algebra. Thus we get the following theorems.

THEOREM 4.1. *Let L be a finite dimensional lattice. Then the elements of L have unique irredundant decompositions into irreducibles if and only if L is semimodular and u_a/a is distributive for each a.*

THEOREM 4.2. *Let L be a finite dimensional semimodular lattice. Then the number of components in the irredundant decompositions of the elements of L is unique if and only if u_a/a is modular for each a in L.*

In this case the replacement property holds for the irredundant decompositions of an element of L.

It should be observed that the semimodularity of the lattice L is equivalent to the semimodularity of u_a/a for each $a \in L$. For if u_a/a is semimodular for each $a \in L$, then it clearly follows that $a,b \succ a \cap b$ implies $a \cup b \succ a,b$. Thus L is weakly semimodular. But it is well known that for finite dimensional lattices weak semimodularity implies semimodularity. Since distributivity implies semimodularity it follows that L is semimodular if u_a/a is distributive for each $a \in L$.

We shall say that a lattice L has a property P *locally* if u_a/a has the property P for each $a \in L$. Hence L is *locally distributive* if u_a/a is distributive for each $a \in L$ and *locally modular* if u_a/a is modular for each $a \in L$. Then the results given in Theorems 4.1 and 4.2 may be stated as follows:

A finite dimensional lattice has unique irredundant decomposition if and only if it is locally distributive.

A finite dimensional semimodular lattice has replaceable irredundant decompositions if and only if it is locally modular.

Finally we note that for semimodular lattices, local distributivity is equivalent to the property that every modular sublattice is distributive.

5. **Lattices satisfying the ascending chain condition.** In relaxing the requirement of finite dimensionality it is most natural to drop the descending chain condition, since by Theorem 3.1 the ascending chain condition alone is sufficient to insure the existence of finite irredundant decompositions into irreducibles. On the other hand, the techniques of §3 for studying irredundant decompositions can no longer be applied since in general covering elements will not exist. However, there is a lattice closely associated with the lattice L in which covering elements always exist; namely, the lattice of dual ideals. A subset A of L is a *dual ideal* if

(1) $a \in A$ and $x \geqq a$ imply $x \in A$;

(2) $a,b \in A$ imply $a \cap b \in A$.

The set of dual ideals of L form a complete lattice in which the join of any set of dual ideals is their set intersection while the meet of any set of dual ideals is the dual ideal generated by their set union. A dual ideal A is *principal* if there exist $a \in L$ such that $A = \{x \in L | x \geqq a\}$ in which case we write $A = (a)$. The principal dual ideals form a sublattice of the lattice of all dual ideals which is isomorphic to the lattice L. The covering theorem asserts that *if $A > (a)$, there exists a dual ideal P such that $A \geqq P \succ (a)$.* Finally we note that when the descending chain condition holds, every dual ideal is principal and hence $P = (p)$ where $p \succ a$ in L.

It is now clear that in case the descending chain condition does not hold, then in place of the quotient lattice u_a/a of L it is appropriate to consider the quotient lattice $U_a/(a)$ of the lattice of dual ideals of L where $U_a = \bigcup\{P | P \succ (a)\}$. In the finite dimensional case u_a/a is always finite dimensional and this property is fundamental in developing the decomposition theory. For lattices satisfying the ascending chain condition, $U_a/(a)$ need not be finite dimensional and this fact is responsible for some of the essential difficulties associated with this more general decomposition theory. However we shall see that $U_a/(a)$ will indeed be finite dimensional in those cases which are relevant to the treatment of uniqueness and replaceability criteria.

The next step in developing the decomposition theory for lattices satisfying only the ascending chain condition is that of formulating an appropriate definition of semimodularity. Perhaps the most natural definition would be to require that the lattice of dual ideals of L be semimodular. However, this requirement is too severe since there exist lattices satisfying the ascending chain condition and having unique irredundant decompositions but for which the lattice of dual ideals is not semimodular. Now it is easy to see that semimodularity for finite dimensional lattices can be put in the form:

$$p \succ a,\ b \geqq a,\ b \ngeqq p \text{ imply } p \cup b \succ b.$$

We shall call an element a of a lattice semimodular if it satisfies the above implication for all p and b. A finite dimensional lattice is thus semimodular if and only if each of its elements is semimodular.

A lattice satisfying the ascending chain condition is then defined to be (upper) semimodular if for each $a \in L$, (a) is semimodular in the lattice of dual ideals of L.

The principal lemmas related to the finite dimensionality of $U_a/(a)$ are the following.

Lemma 5.1. *Let L satisfy the ascending chain condition. If each element of L has a unique irredundant decomposition into irreducibles, then L is semimodular and $U_a/(a)$ is finite dimensional.*

Lemma 5.2. *If $U_a/(a)$ is a Boolean algebra, then it is finite dimensional.*

Lemma 5.3. *Let L be a semimodular lattice satisfying the ascending chain condition. Then if the number of components in the irredundant decompositions of a is unique, $U_a/(a)$ is finite dimensional.*

Lemma 5.4. *Let L be a semimodular lattice satisfying the ascending chain condition. Then if $U_a/(a)$ is modular for each a, $U_a/(a)$ is finite dimensional for each $a \in L$.*

By means of these lemmas, the techniques described in §4 can be applied to arbitrary lattices satisfying the ascending chain condition to give characterizations of lattices with unique decompositions and replaceable decompositions. Analogous to the finite dimensional case let us define L to be locally modular (distributive) if for each $a \in L$ $U_a/(a)$ is modular (distributive). The statement of the fundamental theorems then differ only slightly from those described in §4.

A lattice satisfying the ascending chain condition has unique irredundant decompositions if and only if it is semimodular and locally distributive.

A semimodular lattice satisfying the ascending chain condition has replaceable irredundant decompositions if and only if it is locally modular.

6. **Compactly generated atomic lattices.** In the preceding sections it has been assumed that the lattices under consideration satisfy the ascending chain condition. This means that the decomposition theorems can only be applied to give structure theorems in case the lattice of congruence relations satisfies the ascending chain condition. This is a very restrictive condition for most algebraic systems and hence it is desirable to have decomposition theorems which do not depend upon assumption of a chain condition. On the other hand, it is easy to give examples of lattices having no irreducible elements. For such lattices a decomposition theory is essentially meaningless. Hence some type of restriction is necessary to insure the existence of irreducibles and decompositions into irreducibles. Lattices of congruence relations provide the key to such a restriction in that they always have the property that they are compactly generated. Namely, if $\theta \in \Theta(L)$, then $\theta = \bigcup\{\theta(a,b) | a\ \theta\ b\}$ where $\theta(a,b)$ denotes the congruence relation generated by identifying a and b. Now suppose that $\bigcup\Phi \geqq \theta(a,b)$ for some set Φ of congruence relations ϕ. Then since $a\ \theta(a,b)\ b$ we have $a \bigcup\Phi\ b$ and hence there exist elements $a = x_0, x_1, \cdots, x_n = b$ such that $x_{i-1}\ \phi_i\ x_i$ where $\phi_i \in \Phi$. But then $a(\phi_1 \cup \cdots \cup \phi_n)b$ and hence $\phi_1 \cup \cdots \cup \phi_n \geqq \theta(a,b)$. Thus $\theta(a,b)$ has the property that $\bigcup\Phi \geqq \theta(a,b)$ implies $\bigcup\Phi' \geqq \theta(a,b)$ where Φ' is a finite subset of Φ. An element of a lattice having this property is said to be *compact*. If every element of the lattice is a join of compact elements, the lattice is said to be *compactly generated*. The above argument shows that $\Theta(L)$ is always compactly generated and a similar line of reasoning shows that the lattice of congruence relations on an arbitrary algebraic system is compactly generated.

The property of being compactly generated can be viewed as a generalization of the ascending chain condition. For if the ascending chain condition holds, then the join of an arbitrary set of elements is equal to the join of some finite subset and hence every element of the lattice is compact. Thus the lattice is trivially compactly generated.

At the present time a satisfactory decomposition theory for arbitrary compactly generated lattices does not exist, although in the following section we shall give some indications of the nature of such a general theory. The principal difficulty lies in the fact that irredundant decompositions into irreducibles need not exist. For lattices satisfying the ascending chain condition there exist finite decompositions into irreducibles and hence by refinement there exist irredundant decompositions. Now an element of a compactly generated lattice can always be represented as a meet of irreducibles. For if $a > b$, there exists a compact element c such that $a \geqq c$ and $b \ngeqq c$. Since c is compact there exists a maximal element q such that $q \geqq b$ and $q \ngeqq c$. Then q is completely meet irreducible since if $q = \bigcap X$ where $x > q$ for all $x \in X$ we must have $x \geqq c$ for all $x \in X$ and hence $q = \bigcap X \geqq c$ contrary to $q \ngeqq c$. Thus $q \geqq b$ and $q \ngeqq c$. It follows that each element is the meet of the completely meet irreducibles containing it. This representation is in general