de Gruyter Series in Nonlinear Analysis and Applications 11

Stanislav V. Emelyanov
Sergey K. Korovin
Nikolai A. Bobylev
Alexander V. Bulatov

Homotopy of Extremal Problems

Theory and Applications

Stanislav V. Emelyanov
Sergey K. Korovin
Nikolai A. Bobylev
Alexander V. Bulatov

Homotopy of Extremal Problems

Theory and Applications

Walter de Gruyter · Berlin · New York

Authors

Stanislav V. Emelyanov
Department of Computational Mathematics
and Cybernetics
Moscow State University
Vorob'evy Gory
119899 Moscow
Russia

Sergey K. Korovin
Department of Computational Mathematics
and Cybernetics
Moscow State University
Vorob'evy Gory
119899 Moscow
Russia

Nikolai A. Bobylev
Institute of Control Problems of RAS
65 Profsoyiznaya St.
117997 Moscow
Russia

Alexander V. Bulatov
Institute of Control Problems of RAS
65 Profsoyiznaya St.
117997 Moscow
Russia

Keywords: homotopy method, variational calculus, control theory, nonlinear programming, bifurcation theory, Conley index, nonlinear functional analysis, stability

Library of Congress Cataloging-in-Publication Data

A CIP catalogue record for this book is available from the Library of Congress.

ISBN 978-3-11-018942-1
ISSN 0941-813X

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Printed in Germany.
Cover design: Thomas Bonnie, Hamburg
Editing: Natalia and John Wilson, Oxford, UK
Printing and binding: Hubert & Co. GmbH & Co. KG, Göttingen

de Gruyter Series in Nonlinear Analysis and Applications 11

Preface

This monograph is devoted to the applications of the homotopy method to the investigation of variational problems. The authors have attempted not only to describe applications of the homotopy method to the analysis of general variational problems, but also to include applications to specific problems of analysis, the calculus of variations, mathematical physics, nonlinear programming, etc.

The main constructions of this monograph are based on the following observation: if, when a variational problem is deformed, a critical point remains isolated, and, for some value of the parameter describing the deformation, this critical point is a minimizer, then the critical point is a minimizer for the variational problem for all values of the parameter.

The book consists of an introduction and five chapters.

The first chapter is of an introductory character. It contains information from topology, classical functional analysis, convex and nonsmooth analysis, the theory of differential equations, and the theory of extremal problems.

The second and third chapters are devoted to applications of the homotopy method to the investigation of variational problems. Finite-dimensional problems are studied in the second chapter and infinite-dimensional problems in the third chapter.

Chapter 4 is an exposition of the theory of Conley index. The main results of this chapter are theorems on the homotopy invariance of Conley index.

The final chapter contains a wide variety of applications of the homotopy method. There are applications to problems of classical analysis (proofs of various inequalities, and generalizations and improvements, determination of exact constants, proof of a criterion for quadratic forms to be positive definite), to nonlinear programming problems, to multicriteria problems, to problems of variational calculus and optimal control, to stability theory and to bifurcation theory.

The treatment in the monograph is self-contained. All prerequisite results and definitions of a general character are either given in the first chapter or described when needed. The book is intended to be accessible to beginning graduate students.

The authors are grateful to the scientific editor Academician E.F. Mishchenko for very fruitful discussions and recommendations, to the referees Yu.S. Popkov and E.S. Pyatnitskii for their very helpful suggestions, to V.I. Skalyga for his help in preparing the monograph and to L.A. Selivanova for her highly professional work in designing the book.

The authors are grateful to the editors of de Gruyter, in particular to Dr R. Plato, Prof. J.S. Wilson and Mrs. N. Wilson, for the help in preparing the English translation of the book.

Moscow, June 2007

A.V. Bulatov
S.V. Emelyanov
S.K. Korovin

Table of Contents

Introduction

The homotopy method (or continuation method), which dates back to the nineteenth century, plays an active part today in various branches of mathematics. The general idea is geometrically visual and simple: if we are given an equation (algebraic, differential, integral, integro-differential, operator, etc.) and we want information about its solutions (existence, their local properties, construction of approximate solutions, etc.), then we include this equation in a specially constructed one-parameter family of equations which constitute a homotopy (or deformation) from the equation to some equation which has a known solution, and we "deform this solution with respect to the parameter" to obtain a solution of the original equation. Here is a more formal explanation of this idea.

Suppose that we are given an equation

$$A(x) = 0\,. \tag{1}$$

Assume that we can include Eq. (1) in a one-parameter family

$$A(x;\lambda) = 0 \qquad (0 \leqslant \lambda \leqslant 1) \tag{2}$$

in such a way that Eq. (2) has a solution $x(\lambda)$ which depends smoothly on λ. Suppose that the equation

$$A(x;0) = 0$$

has a solution x_0 and that

$$A(x;1) = A(x)\,.$$

Differentiating the identity

$$A(x(\lambda);\lambda) \equiv 0$$

with respect to λ, we obtain

$$A'_x(x(\lambda);\lambda)\frac{dx(\lambda)}{d\lambda} + A'_\lambda(x(\lambda);\lambda) \equiv 0\,.$$

Thus $x(\lambda)$ is a solution of the Cauchy problem

$$\begin{cases} A'_x(x;\lambda)\dfrac{dx}{dt} + A'_\lambda(x;\lambda) = 0, \\ x(0) = x_0. \end{cases} \tag{3}$$

If this solution can be extended to the interval [0,1], then $x(1)$ is a solution of Eq. (1).

We shall now describe another way to investigate Eq. (2) which is of a discrete character.

We divide the interval $[0, 1]$ into subintervals by choosing points

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = 1 .$$

Let

$$\delta = \max_{0 \leqslant i \leqslant n-1} (\lambda_{i+1} - \lambda_i) .$$

If δ is sufficiently small, then it is reasonable to expect that x_0 is close to a solution $x(\lambda_1)$ of the equation

$$A(x; \lambda_1) = 0 .$$

Taking this as an initial approximation for an iterative procedure (say, Newton's method), we can find, with sufficient accuracy, an approximation of x_1 to $x(\lambda_1)$. We may regard the point x_1, in turn, as an initial condition imposed for the approximate construction of a solution $x(\lambda_2)$ of the equation

$$A(x; \lambda_2) = 0 ,$$

and so on. At the last step we obtain, with the required degree of accuracy, a solution $x(1)$ of Eq. (1).

Of course these procedures have to be justified. Thus, for instance, we need existence theorems for solutions of Eq. (2) for all λ. The assumption that the solution $x(\lambda)$ depends smoothly on λ is rather restrictive since in specific problems the set of solutions of Eq. (2) may turn out to be very complicated. Moreover it is not *a priori* clear how to obtain the one-parameter family of Eqs. (2), although in practice the parameter often λ enters into the equation under study in a natural way. It should be pointed out here that the commonest way to construct a one-parameter family of Eqs. (2) is to take equations of the form

$$\lambda A(x) + (1 - \lambda) B(x) = 0 ,$$

where the standard equation

$$B(x) = 0$$

is constructed using *a priori* information about the equation Eq. (1) under study.

One of the most general and effective ways to apply the homotopy method to the qualitative investigation of operator equations of the form

$$x - C(x) = 0 \tag{4}$$

where C is a completely continuous operator, was developed by Leray and Schauder [159]. In this method, the parameter λ enters into the equations linearly, i.e., the family of equations has the form

$$x - \lambda C(x) = 0 \quad (0 \leqslant \lambda \leqslant 1) . \tag{5}$$

If for all $\lambda \in [0, 1]$ the solutions $x(\lambda)$ of Eq. (5) satisfy a general *a priori* inequality

$$\|x(\lambda)\| \leqslant r \quad (0 \leqslant \lambda \leqslant 1) ,$$

then Eqs. (5) (and, in particular, Eq. (4), the equation of interest) are solvable. The proof of this result is based on a topological invariant introduced by Leray and Schauder, namely, the degree of a mapping. The Leray–Schauder method has been generalized and developed in many theoretical works and also in works of an applied character. Noteworthy here are the works [31, 139, 145, 151, 153, 204, 214] and the bibliography therein.

The homotopy method was used by Gavurin [123] to establish the solvability of operator equations in Banach spaces (cf. also Rosenbloom [195], Polyak [191], Li [161], Zhang De-Tong [229], Wacker [228], Allgower and Georg [5], Smale [216], Hirsch and Smale [133], Chow, Mallet-Paret and Yorke [62], and Kellogg, Li and Yorke [141]).

The homotopy method was apparently first used for the numerical solution of equations by Lahaye [154] (see also [184]). The method was developed in the works by Freudenstein and Roth [121], Shidlovskaya [207], Davidenko [79–83], Roberts and Shipman [194], and Bosarge [48]. For a more extensive bibliography see the monographs [184, 206].

This monograph gives an account of the applications of the homotopy method to variational problems.

1 Preliminaries

This chapter contains the material from functional analysis that is needed in the monograph. No proofs are given since most of the results are well known and can be considered to be classical.

1.1 Topological, Metric, and Normed Spaces

In this section, we introduce the notions of topological, metric, Banach and Hilbert spaces and give some specific examples of spaces which play a role in the monograph.

1.1.1 Topological Spaces

A *topological space* is a set X together with a family $\mathcal{T}$ of subsets of X satisfying the following three conditions:

(1) $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$;
(2) the union of any collection of sets from $\mathcal{T}$ belongs to $\mathcal{T}$;
(3) the intersection of any two sets from $\mathcal{T}$ belongs to $\mathcal{T}$.

A family of subsets satisfying these three conditions is called a *topology* on X. The sets belonging to $\mathcal{T}$ are called *open* sets. A subset F of X is called *closed* if its complement $X \setminus F$ is open. An *open neighborhood* of a subset Y of X is an open set containing Y, and a neighborhood of Y is a set containing an open neighborhood of Y.

Let Y be a subset of X. The family of sets $\{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y, called the *induced* or *subspace* topology on Y.

An *equivalence relation* on a set X is a set of pairs

$$R \subset \{(x, y) : x, y \in X\}$$

satisfying the following conditions:

(1) $(x, x) \in R$ for all $x \in X$;
(2) $(x, y) \in R$ implies $(y, x) \in R$;
(3) $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

If $(x, y) \in R$, then we say that x and y are *equivalent*. The equivalence relation gives rise to a partition of X into pairwise disjoint subsets, called *equivalence classes*, consisting of equivalent elements. The set of equivalence classes corresponding to R is called the *quotient space* of X with respect to R and is denoted by X/R. We write $[x]$ for the equivalence class containing an element x. The *quotient topology* on X/R is defined as follows: a subset $U \subset X/R$ is *open* if the set

$$\bigcup_{[x]\in U} \{x\}$$

is open in X. Let X_1 and X_2 be topological spaces and consider the Cartesian product $X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$. We introduce a topology, called the *product topology* on $X_1 \times X_2$, by taking as open sets all unions of sets of the form $U_1 \times U_2$ with U_1 open in X_1 and U_2 open in X_2.

Let X_1 and X_2 be again two topological spaces. A mapping $f : X_1 \to X_2$ is said to be *continuous* if the complete preimage $f^{-1}(U)$ of any open set $U \subset X_2$ is an open set in X_1. Two mappings f_0, $f_1 : X_1 \to X_2$ are *homotopic* (and we write $f_0 \sim f_1$) if there exists a continuous mapping

$$f : X_1 \times [0,1] \to X_2$$

satisfying the conditions

$$f(x,0) = f_0(x) \quad (x \in X_1)\ ,$$
$$f(x,1) = f_1(x) \quad (x \in X_1)\ .$$

The spaces X_1 and X_2 are *homotopy equivalent* if there exist continuous mappings $f : X_1 \to X_2$ and $g : X_2 \to X_1$ such that the composite mappings $f \circ g$ and $g \circ f$ are homotopic to the respective identity mappings. We write $X_1 \sim X_2$ to indicate that the spaces X_1 and X_2 are homotopy equivalent.

A *topological pair* is an ordered pair (X, A), where X is a topological space and A is an arbitrary subset of X. A mapping $f : X_1 \to X_2$ is a *mapping of the topological pair* (X_1, A_1) *into the topological pair* (X_2, A_2) (and we write $f : (X_1, A_1) \to (X_2, A_2)$) if $f(A_1) \subset A_2$. There is an obvious notion of homotopy equivalence for topological pairs similar to that introduced above for ordinary spaces. Homotopy equivalence is an equivalence relation on any set of topological pairs.

It is sometimes convenient to consider a topological space X as the topological pair $(X, \varnothing)$. In the theory of Conley index, use is made of topological pairs (X, A) in which A contains just one point; topological pairs of this kind are called *topological spaces with base point* (or *pointed spaces*).

Here are two properties of continuous mappings of topological pairs.

Proposition 1.1.1. *Consider mappings* f_0, $f_1 : (X_1, A_1) \to (X_2, A_2)$, g_0, $g_1 : (X_2, A_2) \to (X_3, A_3)$. *If* $f_0 \sim f_1$ *and* $g_0 \sim g_1$, *then* $g_0 \circ f_0 \sim g_1 \circ f_1$.

Proposition 1.1.2. *If the composite mappings* $g \circ f$ *and* $h \circ g$ *in the sequence of mappings*

$$(X_1, A_1) \xrightarrow{f} (X_2, A_2) \xrightarrow{g} (X_3, A_3) \xrightarrow{h} (X_4, A_4)$$