Series on Advances in Statistical Mechanics – Volume 5

POTTS MODELS and RELATED PROBLEMS in STATISTICAL MECHANICS

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SERIES ON ADVANCES IN STATISTICAL MECHANICS

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Preface

This book is largely a rearrangement of a set of lectures I gave at RIMS, Kyoto in spring 1989. The additional material is either from lectures I *attended* in that period, or arose in the process of rearrangement. The book itself was written at Birmingham University Mathematics Department (summer 89) and City University Mathematics Department (autumn 89 onward).

The central theme of Potts models was chosen with the aim of drawing together, in a systematic way, some key ideas from the continuing explosion of statistical mechanical research. The acid test of relevance to this well established physical picture is one sure way to force a coherent direction on the general explosion of results.

The potential drawback of writing a book on such a vital subject is that one feels more and more pressure each week to incorporate the very latest results. Succumbing to this pressure would obviously result in a rather slowly converging process. In any case, most up to the minute research turns out not, on mature reflection, to be suitable for a textbook. I therefore introduced a cut-off on new material after spring 1989 (except where directly pertinent to the presentation of material already included). At time of going into print no aspect of the subject which has matured since then seems, by its omission, to leave a gap in the presentation.

Of course the subject will soon move on, but the fundamentals will still be fundamental. With this in mind, another aim was to provide a handy work of reference for my own research use, just as Baxter's (1982) book (with a different slant and broader but less recent perspective) has become for so many.

I have tried to make the mathematical development as explicit as is consistent with a finite number of pages. As a physicist I have in particular strived to convey some of the pleasure I have found in learning and applying algebraic techniques to physical problems... and physics is forever trying to teach us algebra.

London, England　　　　　　　　　　　　Paul Martin

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Contents

Chapter 1

Introduction

Why Potts models? Why statistical mechanics?

I do not think there need be any mathematical model faithfully describing the universe as a whole. There are, however, some very beautiful mathematical models for physical phenomena perceived in restricted kinematic regimes. To me, then, a mathematical model which takes as input a description of a system at one length scale, and gives as output a description of this system at a different scale, is particularly exciting. Equilibrium statistical mechanics concerns models of this type.

In addition to this aesthetic consideration I note three major practical motivations for the application of statistical mechanics. One is the common need to determine properties of a physical system on a macroscopic scale from an initial description of the system on a relatively small (microscopic) but finite scale. Another is the discrete approximation of quantum field theory, in which the microscopic scale becomes infinitessimal. The third is the use of these physically supported pictures of statistical mechanics to provide insight into the many glorious mathematical spin-offs from the subject.

The Potts models are a special and easily defined class of statistical mechanical models, as we will see. Nonetheless, they are richly structured enough to illustrate almost every conceivable nuance of the subject. In particular, they are at the centre of the most recent explosion of interest generated by the confluence of conformal field theory, knot theory, quantum groups and integrable systems.

We are fortunate that *all* problems in statistical mechanics seem to be related to Potts models. Fortunate, because this means that a general discussion of the subject can be couched in Potts model terms. These models are invaluable in that they allow a ready understanding of their own basic physical significance and, compared to many of their more purely mathematically motivated counterparts, exhibit a robust insensitivity to boundary conditions away from the critical region. Any new result can be measured soberly against its implications for the Potts models. At the same time they present a great challenge as the most tantalising of unsolved models....

1.1 On layout and objectives

There exist several splendid books and reviews having some overlap of ambit with the present work. Baxter's (1982) book is a particularly fine example, and we will mention many others as we go along. The material which has been covered has been covered well, and there is no point in going over the same ground again here. On the other hand, there is plenty of scope for progress in the same philosophical territory, without duplicating technical details which are already reported so lucidly. With this in mind an exhaustive survey of work in the field of statistical mechanics has been omitted in preference to one which, while self-contained, covers predominantly new ground. The review material necessary for self-containedness has, as far as possible, been given a novel slant. This book is not intended, then, to be an alternative to, or review of, existing works in the field, but rather a companion to them.

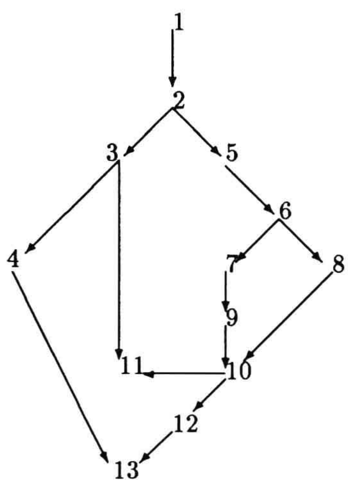
One of the striking features of recent developments in statistical mechanics has been their profound interest to physicists and mathematicians alike. Another feature has been the bewildering proliferation of solvable models and solutions to the star-triangle relations. This Gibbsian/Boltzmannian pulchritude has tended to defy any systematic categorisation. At present, the physical significance of these models is not uniformly clear. However, just as mathematics has been the source of many new models, so the needs and perspectives of physics should provide the main source of organisational criteria.

The first theme introduced here, therefore, will be the basic physical motivation common to these models. This is reviewed in so far as it is needed to understand and organise the models. The basic calculational techniques involved are also introduced, and a framework for understanding and assessing mathematical *results* in physical terms is established. The Potts model is used as an example throughout. This theme might be generally characterised as classical statistical mechanics for mathematicians.

Having established this foundation we then develop an algebraic framework for the statistical mechanical notion of equivalence of models (a key step in categorisation), which leads us to regard the *transfer matrix* as a representative of an element of an associative algebra. It becomes natural, therefore, to characterise and classify such models by the algebra they represent. The development of this characterisation is our second running theme. For example, the most popular such algebra is the Temperley-Lieb algebra, which is the algebraic embodiment of the Potts model. We give a detailed description of this algebra, so that the reader is eventually equipped to go out and construct arbitrary representations, and hence all Temperley-Lieb solutions to the star-triangle relations. This theme can perhaps be characterised as algebra for physicists.

Finally, we discuss some of the ways in which the fruit of the union of ideas coming respectively from the mathematical and physical perspectives may be harvested! This subject is currently the focus of a huge global research drive. The objective here is to get us to the point where we can join in this drive, rather than to attempt to chronicle it fully.

Table of rough interdependence of chapters



Referencing, in a work with a primarily pedagogical ambit, is often a compromise between fastidious acknowledgement of original sources, however obscure, and recommendation of the clearest pedagogical exposition. Here we will not compromise. I apologise in advance to offended parties! The method of referencing is to give names, and dates where necessary to avoid ambiguity, in the text. The full references are then given in the bibliography.

Notations

In statistical mechanics the singleminded persuit of a universally standardised notation has a tendency to become counter productive. It is sensible, nonetheless, to standardise notation as far as possible. There follows a list of standard symbols and notations. Many of them are sufficiently ubiquitous in the literature as to require no explanation, others will be unfamiliar. Anyway, here they are.

$\mathbf{C}$ Field of complex numbers. Unless otherwise stated we will work over the field of complex numbers throughout;

$\mathbf{R}$ Real numbers;

$\mathbf{Z}$ Integers;

$\mathbf{N}, \mathbf{Z}_+$ Natural numbers, positive integers;

Z_Q cyclic group of order Q (i.e. additive version: elements $p = 0, 1, .., Q-1$, composition given by addition mod Q; multiplicative version: elements $\exp(2\pi ip/Q)$);

$[p]$ integer part of $p \in \mathbf{R}$;

d dimension of a physical system, i.e. $d \in \mathbf{N}$;

1_d d dimensional unit matrix;

$$\text{diagonal}(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix};$$

$[A, B]$ $= AB - BA$, commutator of matrices A, B;

$\{A, B\}$ $= AB + BA$, anticommutator of matrices A, B;

$\{U_i \; ; \; i = 1, .., k\}$ a set of k objects U_i, e.g. generators of an algebra;

$\delta_{a,b}$ $= \begin{cases} 1 & \text{if } a = b; \\ 0 & \text{otherwise}; \end{cases}$

V_d d dimensional vector space, or specific set of basis elements;

V vector space of indeterminate dimension, *or* specific set of basis elements for V_d;

$\dim V = d$ order of set V;

$A \otimes B$ tensor or direct product of matrices (or vector spaces) A and B;

$A \oplus B$ direct sum of matrices (or vector spaces) A and B;

$\oplus_i d_i A_i$ $= (1_{d_1} \otimes A_1) \oplus (1_{d_2} \otimes A_2) \oplus ...$, $d_i \in \mathbf{N}$, A_i matrices;

$\otimes^n V_d$ $= V_d \otimes V_d \otimes V_d \otimes ... \otimes V_d$;

S_d simple module of dimension d;

P_d projective module of dimension d;

$End(V_d)$ endomorphisms of V_d;

$B = End_A(V_d)$ A is an algebra, V_d an A-module, then B is the algebra of linear transformations on V_d which commute with the action of A;

$M_d(\mathbf{C})$ algebra of d-dimensional matrices over the complex numbers $\mathbf{C}$;

N Number of particles in system;

$\{\sigma\}$ a set of variables $\{\sigma_i; i = 1, .., N\}$;

$\hom(\{\sigma\}, V)$ hom-set consisiting of all functions on $\{\sigma\}$ to set V, i.e. $\hom(\{\sigma\}, V) \sim V^N$;

$\{\sigma_i\}$ a possible *configuration* of a set of variables $\sigma_i \in V$, i.e. $\{\sigma_i\} \in hom(\{\sigma\}, V)$;

H Hamiltonian;

Z Partition function;

$< O >$ Expectation value of observable O;

$\langle ij \rangle$ nearest neighbour pair of lattice sites, i and j;