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Essential MATHEMATICAL Methods for Physicists

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Vector Identities

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}, \quad A^2 = A_x^2 + A_y^2 + A_z^2, \quad \mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{\mathbf{x}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\hat{\mathbf{y}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{\mathbf{z}}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = C_x\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - C_y\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + C_z\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,\mathbf{A}\cdot\mathbf{C} - \mathbf{C}\,\mathbf{A}\cdot\mathbf{B}, \quad \sum_k \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Vector Calculus

$$\mathbf{F} = -\boldsymbol{\nabla}V(r) = -\frac{\mathbf{r}}{r}\frac{dV}{dr} = -\hat{\mathbf{r}}\frac{dV}{dr}, \quad \boldsymbol{\nabla}\cdot(\mathbf{r}f(r)) = 3f(r) + r\frac{df}{dr},$$

$$\boldsymbol{\nabla}\cdot(\mathbf{r}r^{n-1}) = (n+2)r^{n-1}$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$$

$$\boldsymbol{\nabla}\cdot(S\mathbf{A}) = \boldsymbol{\nabla}S\cdot\mathbf{A} + S\boldsymbol{\nabla}\cdot\mathbf{A}, \quad \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0, \quad \boldsymbol{\nabla}\times(S\mathbf{A}) = \boldsymbol{\nabla}S\times\mathbf{A} + S\boldsymbol{\nabla}\times\mathbf{A}, \quad \boldsymbol{\nabla}\times(\mathbf{r}f(r)) = 0,$$

$$\boldsymbol{\nabla}\times\mathbf{r} = 0$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,\boldsymbol{\nabla}\cdot\mathbf{B} - \mathbf{B}\,\boldsymbol{\nabla}\cdot\mathbf{A} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B},$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}$$

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{B}\,d^3r = \int_S \mathbf{B}\cdot d\mathbf{a}, \quad \text{(Gauss)}, \quad \int_S (\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}, \quad \text{(Stokes)}$$

$$\int_V (\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi)\,d^3r = \int_S (\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi)\cdot d\mathbf{a}, \quad \text{(Green)}$$

$$\boldsymbol{\nabla}^2\frac{1}{r} = -4\pi\delta(\mathbf{r}), \quad \delta(ax) = \frac{1}{|a|}\delta(x), \quad \delta(f(x)) = \sum_{i,\, f(x_i)=0,\, f'(x_i)\neq 0} \frac{\delta(x-x_i)}{|f'(x_i)|},$$

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}\,d\omega, \quad \delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}},$$

$$\delta(x-t) = \sum_{n=0}^{\infty}\varphi_n^*(x)\varphi_n(t)$$

Curved Orthogonal Coordinates

Cylinder Coordinates

$$q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z; \quad h_1 = h_\rho = 1,\ h_2 = h_\varphi = \rho,\ h_3 = h_z = 1,$$

$$\mathbf{r} = \hat{\mathbf{x}}\rho\cos\varphi + \hat{\mathbf{y}}\rho\sin\varphi + z\hat{\mathbf{z}}$$

Spherical Polar Coordinates

$$q_1 = r,\ q_2 = \theta,\ q_3 = \varphi; h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_\varphi = r\sin\theta,$$

$$\mathbf{r} = \hat{\mathbf{x}}r\sin\theta\cos\varphi + \hat{\mathbf{y}}r\sin\theta\sin\varphi + \hat{\mathbf{z}}r\cos\theta$$

$$d\mathbf{r} = \sum_i h_i dq_i \hat{\mathbf{q}}_i, \quad \mathbf{A} = \sum_i A_i\hat{\mathbf{q}}_i, \quad \mathbf{A}\cdot\mathbf{B} = \sum_i A_iB_i, \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\int_V f\, d^3r = \int f(q_1, q_2, q_3) h_1h_2h_3\, dq_1\, dq_2\, dq_3 \qquad \int_L \mathbf{F}\cdot d\mathbf{r} = \sum_i \int F_i h_i\, dq_i$$

$$\int_S \mathbf{B}\cdot d\mathbf{a} = \int B_1h_2h_3\, dq_2\, dq_3 + \int B_2h_1h_3\, dq_1dq_3 + \int B_3h_1h_2\, dq_1dq_2,$$

$$\nabla V = \sum_i \hat{\mathbf{q}}_i \frac{1}{h_i}\frac{\partial V}{\partial q_i},$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(F_1h_2h_3) + \frac{\partial}{\partial q_2}(F_2h_1h_3) + \frac{\partial}{\partial q_3}(F_3h_1h_2)\right]$$

$$\nabla^2 V = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_2h_1}{h_3}\frac{\partial V}{\partial q_3}\right)\right]$$

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{q}}_1 & h_2\hat{\mathbf{q}}_2 & h_3\hat{\mathbf{q}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

Mathematical Constants

$$e = 2.718281828, \quad \pi = 3.14159265, \quad \ln 10 = 2.302585093,$$

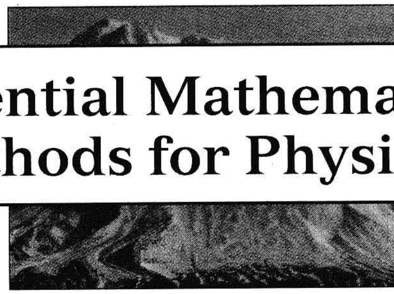
$$1\text{ rad} = 57.29577951^\circ, \quad 1^\circ = 0.0174532925\ \text{rad},$$

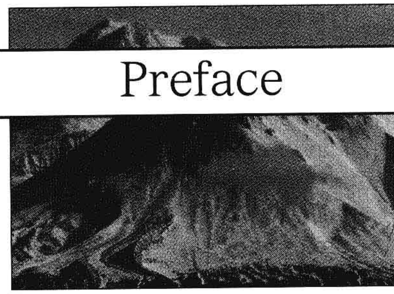
$$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)\right] = 0.577215661901532$$

(Euler-Mascheroni number)

$$B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = B_8 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \ldots$$ (Bernoulli numbers)

Essential Mathematical Methods for Physicists





Preface

This text is designed for the usual introductory physics courses to prepare undergraduate students for the level of mathematics expected in more advanced undergraduate physics and engineering courses. One of its goals is to guide the student in learning the mathematical language physicists use by leading them through worked examples and then practicing problems. The pedagogy is that of introducing concepts, designing and refining methods, and practicing them repeatedly in physics examples and problems. Geometric and algebraic approaches and methods are included and are more or less emphasized in a variety of settings to accommodate different learning styles of students. Sometimes examples are solved in more than one way. Theorems are usually derived sketching the underlying ideas and describing the relevant mathematical relations so that one can recognize the assumptions they are based on and their limitations. These proofs are not rigorous in the sense of the professional mathematician, and no attempt was made to formulate theorems in their most general form or under the least restrictive assumptions.

An important objective of this text is to train the student to formulate physical phenomena in mathematical language, starting from intuitive and qualitative ideas. The examples in the text have been worked out so as to develop the mathematical treatment along with the physical intuition. A precise mathematical formulation of physical phenomena and problems is always the ultimate goal.

Text Overview

In Chapter 1 the basic concepts of vector algebra and vector analysis are introduced and applied to classical mechanics and electrodynamics. Chapter 2 deals with the extension of vector algebra and analysis to curved orthogonal coordinates, again with applications from classical mechanics and electrodynamics. These chapters lay the foundations for differential equations in Chapters 8, 9, and 16; variational calculus in Chapter 18; and nonlinear analysis in Chapter 19. Chapter 3 extends high school algebra of one or two linear equations to determinants and matrix solutions of general systems of linear equations, eigenvalues and eigenvectors, and linear transformations in real and complex vector spaces. These chapters are extended to function spaces of solutions of differential equations in Chapter 9, thereby laying the mathematical foundations for and formulation of quantum mechanics. Chapter 4 on group theory is an introduction to the important concept of symmetry in modern physics. Chapter 5 gives a fairly extensive treatment of series that form the basis for the special functions discussed in Chapters 10–13 and also complex functions discussed in Chapters 6 and 7. Chapter 17 on probability and statistics is basic for the experimentally oriented physicist. Some of its content can be studied immediately after completion of Chapters 1 and 2, but later sections are based on Chapters 8 and 10. Chapter 19 on nonlinear methods can be studied immediately after completion of Chapter 8, and it complements and extends Chapter 8 in many directions. Chapters 10–13 on special functions contain many examples of physics problems requiring solutions of differential equations that can also be incorporated in Chapters 8 and 16. Chapters 14 and 15 on Fourier analysis are indispensible for a more advanced treatment of partial differential equations in Chapter 16.

Historical remarks are included that detail some physicists and mathematicians who introduced the ideas and methods that later generations perfected to the tools we now use routinely. We hope they provide motivation for students and generate some appreciation of the effort, devotion, and courage of past and present scientists.

Pathways through the Material

Because the text contains more than enough material for a two-semester undergraduate course, the instructor may select topics to suit the particular level of the class. Chapters 1–3 and 5–8 provide a basis for a one-semester course in mathematical physics. By omitting some topics, such as symmetries and group theory and tensors, it is possible in a one-semester course to also include parts of Chapters 10–13 on special functions, Chapters 14 and 15 on Fourier analysis, Chapter 17 on probability and statistics, Chapter 18 on variational calculus, or Chapter 19 on nonlinear methods.

A two-semester course can treat tensors and symmetries in Chapters 2 and 4 and special functions in Chapters 10–13 more extensively, as well as variational calculus in Chapter 18 in support of classical and quantum mechanics.

Problem-Solving Skills

Students should study the text until they are sure they understand the physical interpretation, can derive equations with the book closed, can make predictions in special cases, and can recognize the limits of applicability of the theories and equations of physics. However, physics and engineering courses routinely demand an even higher level of understanding involving active learning in which students can apply the material to solve problems because it is common knowledge that we only learn the mathematical language that physicists use by repeatedly solving problems.

The problem sets at the end of sections and chapters are arranged in the order in which the material is covered in the text. A sufficient variety and level of difficulty of problems are provided to ensure that anyone who conscientiously solves them has mastered the material in the text beyond mere understanding of step-by-step derivations. More difficult problems that require some modification of routine methods are also included in various sets to engage the creative powers of the student, a skill that is expected of the professional physicist.

Computer Software

Problems in the text that can be solved analytically can also be solved by modern symbolic computer software, such as Macsyma, Mathcad, Maples, Mathematica, and Reduce, because these programs include the routine methods of mathematical physics texts. Once the student has developed an analytical result, these powerful programs are useful for checking and plotting the results. Finding an analytical solution by computer without understanding how it is derived is pointless. When computers are used too early for solving a problem, many instructors have found that students can be led astray by the computers. The available computer software is so diverse as to preclude any detailed discussion of it. Each instructor willing to make use of computers in the course will have to make a choice of a particular software and provide an introduction for the students. Many problems and examples in the text may then be adapted to it. However, their real utility and power lie in the graphics software they include and the ability to solve problems approximately and numerically that do not allow for an analytical solution. Special training is needed, and the text can be used to train students in approximation methods, such as series and asymptotic expansions, or integral representations that are suitable for further symbolic computer manipulations.

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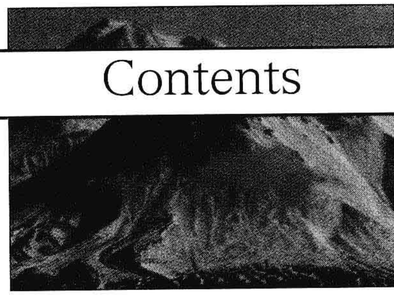

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