New Syllabus Mathematics for O-Level 1

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Preface

These two volumes are intended for students who want to pass O-Level mathematics in the modern syllabus. They are particularly suitable for those who need to follow a thorough revision course, whether at school or as full-time, day-release or evening students at colleges of further education. Since the only mathematical knowledge assumed is simple arithmetic, the books are also suitable for those who need a pass in O-Level mathematics to improve their promotion prospects, and are starting the modern syllabus for the first time.

The majority of the exercises are divided into A and B sections. The questions in the A sections are generally shorter and intended for routine practice in the techniques appropriate to each part of the text. Longer and more thought-provoking questions are found in the B sections. Each of the sixteen chapters ends with a multiple-choice test and a selection of miscellaneous examples from past examination papers.

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The text covers the 'modern' alternative syllabus of each of the major examining boards, and the authors acknowledge with thanks the permission given by the boards to quote examination questions. The source of each question is shown in the text by the following abbreviations

(AEB)	Associated Examining Board
(C)	University of Cambridge Local Examinations Syndicate
(L)	University of London, University Entrance and School Examinations Council
(JMB)	Joint Matriculation Board
(NI)	Northern Ireland Schools Examinations Council
(O)	Oxford Delegacy of Local Examinations
(OC SMP)	Oxford and Cambridge Schools Examination Board. Schools Mathematics Project
(S)	Southern Universities Joint Board
(SCE)	Scottish Certificate of Education Examination Board.

Notation

$\{a, b, c, \ldots\}$	the set of $a, b, c, \ldots$
$:$	such that
$\in$	is an element of
$\notin$	is not an element of
$n(\ )$	the number of elements in the set of
$\varnothing$	the empty (null) set
$\mathscr{E}$	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a subset of
A'	the complement of the set A
N	the set of natural numbers
Z	the set of integers
R	the set of real numbers
PQ	operation Q followed by operation P
$f{:}x \rightarrow y$	the function of mapping the set X into the set Y
$f(x)$	the image of x under the function f
f^{-1}	the inverse of the function f
fg	the function f of the function g
—○—○—	open interval on the number line
—●—●—	closed interval on the number line
$\{x: -2 < x < 7\}$	the set of values of x such that . . .
$\Rightarrow$	implies that
$\Leftarrow$	is implied by
$\Leftrightarrow$	implies and is implied by
$=$	is equal to
$\equiv$	is identically equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$\nless$	is not less than
$\ngtr$	is not greater than
$\mid\ \mid$	the unsigned part of a signed number, that is the modulus
∞	infinity
M'	the transpose of the matrix M

Contents

1 Number Systems

The two essentials of education are literacy and numeracy. To be literate requires a knowledge of the letters of the alphabet, ordering these into words and constructing sentences according to the rules of grammar; to be numerate requires a knowledge of numbers, order relations between the numbers, and operations defined on the numbers according to certain laws. In this chapter the four different kinds of numbers are defined, and the three laws which govern the basic operations of arithmetic are introduced; addition and subtraction, multiplication and division.

1.1 Natural Numbers

The counting numbers are called natural numbers. In the United Kingdom, as in most countries, the denary system of counting in tens is used, and anyone starting a mathematics course for O-level must be familiar with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 as well as the addition and multiplication tables associated with them. No other prior knowledge will be assumed except the ordinary processes of arithmetic on the natural numbers including long multiplication and division.

Multiples and Factors

An even number is a multiple of 2 and the first even number is 2×1, the next is 2×2, the third 2×3 and so on. Similarly, 14 is 7×2, 21 is 7×3, and so 14 and 21 are both multiples of 7.

The converse of the statement '14 is a multiple of 7' is '7 is a factor of 14', or '14 has 7 as a factor'.

Prime Numbers

Every number has 1 as a factor and a natural number which has no factors except itself and 1 is called a prime number. The primes up to 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Some mathematicians enjoy finding prime numbers, and it has been proved that they form an infinite set. The largest one known up to 1971 is said to have 6002 digits.

Example 1.1

Express as a product of prime factors (a) 35, (b) 90, (c) 144.

(a) $35 = 5 \times 7$
(b) $90 = 2 \times 3 \times 3 \times 5$
(c) $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$

Highest Common Factor

The highest common factor (H.C.F.) of a given set of numbers is the greatest number which is a factor of each of the given numbers. Thus the H.C.F. of 15, 27, 39 is 3, since

$15 = 3 \times 5$, $27 = 3 \times 3 \times 3$, and $39 = 3 \times 13$

3 is the greatest number which is a factor of each.

Example 1.2

Find the H.C.F. of 42, 56, 126.

Expressing each number as a product of primes

$$42 = 2 \times 3 \times 7$$
$$56 = 2 \times 2 \times 2 \times 7$$
$$126 = 2 \times 3 \times 3 \times 7$$

The primes which are factors of all three numbers are 2 and 7 therefore the H.C.F. is $2 \times 7 = \underline{14}$.

Least Common Multiple

The least common multiple (L.C.M.) of a set of numbers is the lowest number which contains as a factor each of the given numbers. Thus the L.C.M. of 5, 6, and 10 is 30, since there is no number lower than 30 which is a multiple of 5, of 6, and of 10.

Example 1.3

Find the L.C.M. of 15, 28, 140.

$15 = 3 \times 5$

$28 = 2 \times 2 \times 7$

$140 = 2 \times 2 \times 5 \times 7$

The L.C.M. is $3 \times 5 \times 2 \times 2 \times 7 = \underline{420}$.

Example 1.4

Find the L.C.M. of 9, 24, 30.

$9 = 3 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$30 = 2 \times 3 \times 5$

The L.C.M. is $3 \times 3 \times 2 \times 2 \times 2 \times 5 = \underline{360}$.

Exercise 1.1

In each case find the lowest number into which the given numbers divide (L.C.M.).

(1) 8, 16 (2) 6, 8 (3) 5, 7 (4) 3, 5, 15
(5) 3, 7, 12 (6) 8, 18, 21 (7) 4, 5, 10
(8) 6, 22, 121.

Find the highest common factor (H.C.F.) of

(9) 20, 24, 28 (10) 6, 24, 144 (11) 80, 96, 272
(12) 84, 126, 196.

1.2 Counting in Bases other than Ten

In binary arithmetic, base two, there are only two digits, 0 and 1; in base three there are three digits, 0, 1 and 2; and in base four there are four digits, 0, 1, 2, 3.

Index Notation and Place Holding

In base ten, one hundred is equal to ten tens, and one thousand is ten hundreds. In index notation this is written as

$$100 = 10 \times 10 = 10^2$$

$$1000 = 10 \times 100 = 10 \times 10 \times 10 = 10^3$$

When we write a number such as 7604 we mean 7 thousands, 6 hundreds, no tens and 4 units, and in index notation this would be

10^4	10^3	10^2	10^1	10^0
	7	6	0	4

where 10^0 is defined as 1. Moving a digit to the next column on the left has the effect of multiplying it by ten, so noughts are inserted as place holders.

A similar system is used for other bases, for example in the binary scale the columns are powers of two.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	0	1	1	0

The binary number 1100110 is equivalent to $64 + 32 + 4 + 2 = 102$.

The powers (indices) are always written in base ten, and the units column has index 0 in every base.

A study of the following table of numbers written in different bases shows how they compare.

	Base								
	10	9	8	7	6	5	4	3	2
Number	1	1	1	1	1	1	1	1	1
	2	2	2	2	2	2	2	2	10
	3	3	3	3	3	3	3	10	11
	4	4	4	4	4	4	10	11	100
	5	5	5	5	5	10	11	12	101
	6	6	6	6	10	11	12	20	110
	7	7	7	10	11	12	13	21	111
	8	8	10	11	12	13	20	22	1000
	9	10	11	12	13	14	21	100	1001
	10	11	12	13	14	20	22	101	1010
	15	16	17	21	23	30	33	120	1111
	20	22	24	26	32	40	110	202	10100
	30	33	36	42	50	110	132	1010	11110
	50	55	62	101	122	200	302	1212	110010
	75	83	113	135	203	300	1023	2210	1001011
	100	121	144	202	244	400	1210	10201	1100100

Suffix Notation

When using a base other than ten, the base is indicated by a suffix after the number. For example $144_5 = 49$, means 144 in base 5 is equal to 49 in base 10.

By convention the suffix is always written in base ten, so that although the digit 5 does not appear in a base five number the meaning of 144_5 is unambiguous.

Bases higher than ten can be used, but new symbols are needed for the digits representing ten, eleven, twelve, etc., and there is no generally accepted system.

Example 1.5

What is the equivalent in base 10 of the following numbers? (a) 111111_2 (b) 2112_3 (c) 14312_5.

(a) The powers of 2 are

2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1
1	1	1	1	1	1

$111111_2 = 32 + 16 + 8 + 4 + 2 + 1 = \underline{63}$

(b) The powers of 3 are

3^5	3^4	3^3	3^2	3^1	3^0
243	81	27	9	3	1
		2	1	1	2

$2112_3 = 2 \times 27 + 9 + 3 + 2 = \underline{68}$

(c) The powers of 5 are

5^5	5^4	5^3	5^2	5^1	5^0
3125	625	125	25	5	1
	1	4	3	1	2

$14312_5 = 625 + 500 + 75 + 5 + 2 = \underline{1207}$

Example 1.6

Find the value of x for which (a) $x_4 = 67_8$ and (b) $23_x = 12_9$.

(a) $67_8 = 6 \times 8 + 7 = 55_{10}$

We require the equivalent in base 4 of 55_{10}. The powers of 4 are

4^3	4^2	4^1	4^0
64	16	4	1
	3	1	3

$55_{10} = 3 \times 16 + 1 \times 4 + 3 = 313_4$

$313_4 = 67_8$

Therefore $x = \underline{313}$

(b) $12_9 = 9 + 2 = 11_{10}$

$23_x = 11_{10}$

$2 \times x + 3 = 8 + 3$

$2 \times x = 8$

Therefore $x = \underline{4}$

Exercise 1.2

(1) Express the following numbers in the denary scale.

(a) 101_2 (b) 235_6 (c) 10110_2 (d) 10211_3 (e) 1234_5 (f) 2102_8 (g) 225_6 (h) 3000_4 (i) 200_7 (j) 110101_2 (k) 20202_3

(2) Write the denary numbers in the base shown in square brackets.

(a) 9[2] (b) 18[3] (c) 27[5] (d) 39[3] (e) 23[2] (f) 85[4] (g) 35[2] (h) 80[8]

(3) Find, in each case, the value of x.

(a) $x_3 = 236_7$ (b) $x_2 = 200_3$ (c) $x_6 = 674_8$ (d) $15_x = 13_9$ (e) $101011_2 = 111_x$

Changing Base by Successive Division

Calculating the powers of different numbers and setting them out in columns can be a tedious process when there are several digits, and so to convert from base 10 to another base a simpler method, successive division, is used.

Example 1.7

(a) Change 8973 to base 7. (b) Change 547 to base 4.

(a) The base 10 number is divided repeatedly by 7 until the quotient becomes zero, noting the remainder at each step.

7)8973	
7)1281	6 ↑
7) 183	0
7) 26	1
7) 3	5
0	3

The number in base 7 is obtained by taking the remainders in order, the units figure being at the top.

$\underline{8973_{10} = 35106_7}$

(b)

4)547	
4)136	3 ↑
4) 34	0
4) 8	2
4) 2	0
0	2

$\underline{547_{10} = 20203_4}$

Doublets and Triads

There is a special method which can be used for changing bases when one of the bases concerned is a power of the other, such as 8 and 2, 4 and 2, 9 and 3.

Example 1.8

(a) Express the octal number 1777 as a binary number. (b) Change 11100110111_2 to base 8.

(a) An octal number is in base 8, and each digit in a base 8 number is equivalent to a triad of digits in base 2, for example $5_8 = 101_2$, $7_8 = 111_2$.

To convert to a binary number each separate digit is replaced by a triad, extra noughts on the left being deleted at the end.

Thus $1777_8 = 001111111111$
$= \underline{1111111111_2}$

(b) Changing from binary to octal, the binary number is counted in triads, starting at the right.

$11100110111_2 = \underline{3467_8}$

Example 1.9

(a) Change the binary number 11011010001 to base 4. (b) Express 2584_9 in base 3.

(a) $4 = 2^2$, and each pair of digits, called a doublet, in base 2 is replaced by a single digit in base 4, starting at the right.

$11011010001_2 = \underline{123101_4}$

(b) $9 = 3^2$, and so every digit in a base 9 number can be replaced by a doublet in base 3.

$2584_9 = \underline{2122211_3}$

When changing from a denary to a binary number, much time is saved by converting first to base 8 by successive division, and then using triads to change from the octal to the binary number. Similarly, to change a denary number to base 3, it is quicker to change to base 9 and then use doublets to write down the base 3 equivalent. Computer programmers use base 8 when devising programs to test the computers.

Example 1.10

Write the denary number 1977 in (a) base 2 and (b) base 3.

(a) Change to base 8

$$\begin{array}{r|l r}
8 & 1977 & \\
8 & 247 & 1 \uparrow \\
8 & 30 & 7 \\
8 & 3 & 6 \\
 & 0 & 3
\end{array}$$

$1977_{10} = 3671_8$
$= \underline{11110111001_2}$

(b) Change to base 9

$$\begin{array}{r|l r}
9 & 1977 & \\
9 & 219 & 6 \uparrow \\
9 & 24 & 3 \\
9 & 2 & 6 \\
 & 0 & 2
\end{array}$$

$1977_{10} = 2636_9$
$= \underline{2201020_3}$

Addition and Subtraction in Bases other than Ten

The processes of arithmetic are the same whatever base is used, but when 'carrying one', the 'one' takes the value of the base being used.

For example, in base 10 we say $6+6 = 12\,(10+2)$ but in base 8 it would be $6+6 = 14\ (8+4)$ 'one carried' is 8 in base 8.

Similarly, in base 6, $4+4 = 12\,(6+2)$ but in base 5, $4+4 = 13\ (5+3)$.

A few simple examples are worked here to illustrate the method, and when they have been studied carefully, it is recommended that the following exercises should be worked through until you are able to do simple calculations in any base.

Example 1.11

Base 2

$1+1=10,\ 1+1+1=11,\ 10-1=1,\ 11-1=10$

$$\begin{array}{r} 1101 \\ +\ \ 101 \\ \hline 10010 \end{array} \qquad \begin{array}{r} 111001 \\ +\ \ 11001 \\ \hline 1010010 \end{array} \qquad \begin{array}{r} 10011 \\ 1010 \\ +\ \ 101 \\ \hline 100010 \end{array}$$

$$\begin{array}{r} 111 \\ -\ \ 10 \\ \hline 101 \end{array} \qquad \begin{array}{r} 100011 \\ -\ \ 1001 \\ \hline 11010 \end{array} \qquad \begin{array}{r} 1101001 \\ -\ \ 1111 \\ \hline 1011010 \end{array}$$

Example 1.12

Base 5

$3+3 = 11,\quad 4+3 = 12,\quad 4+4 = 13$

$$\begin{array}{r} 30420 \\ +10403 \\ \hline 41323 \end{array} \qquad \begin{array}{r} 44444 \\ +\ \ 3112 \\ \hline 103111 \end{array} \qquad \begin{array}{r} 3431 \\ -\ \ 113 \\ \hline 3313 \end{array} \qquad \begin{array}{r} 41012 \\ -\ \ 3344 \\ \hline 32113 \end{array}$$

Example 1.13

Base 8

$7+1=10,\quad 6+5=13,\quad 12-3=7,\quad 25-7=16$

$$\begin{array}{r} 7657 \\ +\ \ 634 \\ \hline 10513 \end{array} \qquad \begin{array}{r} 17635 \\ +45234 \\ \hline 65071 \end{array} \qquad \begin{array}{r} 31642 \\ -20476 \\ \hline 11144 \end{array} \qquad \begin{array}{r} 2453 \\ -1675 \\ \hline 556 \end{array}$$

Exercise 1.3

(1) Evaluate

(a) $11_2 + 11_2$ (b) $23_4 + 32_4$ (c) $11_3 - 2_3$
(d) $5_6 + 4_6$ (e) $64_7 - 6_7$.

(2) Evaluate, leaving your answer in the same base as the question

(a) $101_2 + 111_2$ (b) $22_5 + 13_5$ (c) $21_3 - 12_3$
(d) $222_5 - 34_5$ (e) $324_7 - 35_7$ (f) $1011_2 - 111_2$.

(3) Evaluate

(a) $101101_2 + 1011_2 - 10110_2$
(b) $11022_3 + 222_3 - 1212_3$.

(4) Convert
(a) 7651_{10} to base 7
(b) 925_{10} to base 5
(c) 284_{10} to base 2.
(5) Write down the smallest denary number with three digits and convert it to base 2.
(6) State the largest denary number with two digits and convert this number to base 5.
(7) Find the value of x when
(a) $111_2 + 111_2 = 24_x$ (b) $323_5 - 124_5 = 100_x$.
(8) If $323_x + 234_x = 1112_x$, find the value of x.
(9) Express
(a) 666_8 as a binary number
(b) 212212_3 in base 9
(c) 10110110_2 in base 8
(d) 11110101_2 in base 4.

Multiplication and Division in Bases other than Ten

When working with low base numbers even simple calculations need long multiplication and division, and so addition and subtraction should be mastered before this section is attempted.

Example 1.14

Base 3

$$\begin{array}{r} 121 \\ \times \quad 2 \\ \hline 1012 \\ \hline \end{array} \qquad \begin{array}{r} 1221 \\ \times \quad 12 \\ \hline 10212 \\ 12210 \\ \hline 100122 \\ \hline \end{array} \qquad \begin{array}{r} 2222 \\ \times \quad 222 \\ \hline 12221 \\ 122210 \\ 1222100 \\ \hline 2212001 \\ \hline \end{array} \qquad \begin{array}{r} 1222 \\ 211\overline{)1121012} \\ 211 \\ \hline 2100 \\ 1122 \\ \hline 2011 \\ 1122 \\ \hline 1122 \\ 1122 \\ \hline \end{array}$$

Example 1.15

Base 6

$5 \times 3 = 23$, $4 \times 4 = 24$, $45 \times 3 = 223$

$$\begin{array}{r} 33 \\ \times \quad 21 \\ \hline 33 \\ 1100 \\ \hline 1133 \\ \hline \end{array} \qquad \begin{array}{r} 3345 \\ \times \quad 144 \\ \hline 22312 \\ 223120 \\ 334500 \\ \hline 1024332 \\ \hline \end{array} \qquad \begin{array}{r} 3241 \\ 42\overline{)225402} \\ 210 \\ \hline 154 \\ 124 \\ \hline 300 \\ 252 \\ \hline 42 \\ 42 \\ \hline \end{array} \qquad \begin{array}{r} 243 \\ 504\overline{)220200} \\ 1412 \\ \hline 3500 \\ 3224 \\ \hline 2320 \\ 2320 \\ \hline \end{array}$$

Multiplication and Division by a Power of the Base Number

Multiplying by 10 in any base is effected by moving every digit to the next column on the left, since it increases the index by one. Similarly, multiplying by 100 increases the index by two and so moves each digit two columns to the left. To divide by 10 move each digit to the next column on the right.

Thus $11_2 \times 10_2 = 110_2$, $212_3 \times 100_3 = 21200_3$
$3230_4 \div 10_4 = 323_4$, $111000_2 \div 100_2 = 1110_2$

Prime Numbers and Factors

A prime number remains prime however it is written, and if one number is a factor of another in one base it remains a factor when the base is changed.

Example 1.16

Express as a product of prime factors (a) 1320_5 and (b) 1665_7.

(a) $1320_5 = 10 \times 132$
$= 10 \times 3 \times 24$
$= \underline{10 \times 3 \times 2 \times 12}$ $(12_5 = 7_{10})$

(b) $1665_7 = 2 \times 666$
$= 2 \times 2 \times 333$
$= 2 \times 2 \times 3 \times 111$
$= \underline{2 \times 2 \times 3 \times 3 \times 25}$ $(25_7 = 19_{10})$

Example 1.17

What are the prime factors of the following numbers?

(a) 2221_3 (b) 3231_4 (c) 2362_8

(a) $2221_3 = \underline{79_{10}}$ which is prime

(b) $3231_4 = \underline{3 \times 1033}$ $(1033_4 = 79_{10})$

(c) $2362_8 = 2 \times 1171$
$= \underline{2 \times 3 \times 323}$ $(323_8 = 211_{10}$ which is prime$)$

Exercise 1.4

A

Evaluate

(1) $21_3 \times 2_3$ (2) $13_5 \times 4_5$ (3) $54_6 \times 5_6$
(4) $12_3 \times 2_3$ (5) $37_8 \times 6_8$ (6) $101_3 \div 2_3$
(7) $2301_4 \div 3_4$ (8) $3333_5 \div 4_5$
(9) $221_6 \div 5_6$ (10) $101_2 \times 10_2$
(11) $312_4 \times 100_4$ (12) $1120_3 \div 10_3$
(13) $5500_6 \div 100_6$.

B

(1) Evaluate

(a) $11011_2 \times 111_2$ (b) $213_4 \times 23_4$
(c) $212_3 \times 121_3$ (d) $1001011_2 \div 101_2$
(e) $20111_3 \div 21_3$ (f) $32131_5 \div 34_5$.

(2) Write the following as a product of prime numbers in the given base.
(a) 11110_2 (b) 330_4 (c) 432_5.

(3) Calculate the number midway between 1011_2 and 11011_2.

(4) The average of the three numbers 212_3, 122_3 and 222_3 is found by dividing their sum by 3_{10}. Find this average.

(5) If $4_n \times 412_n = 2251_n$, find the base number n.

(6) Find the value in the binary scale of $2 \times 2 \times 2 \times 2 + 2$.

(7) What is the highest number in base 3 that will divide into each of the numbers 220_3, 110_3, 22_3? Also, find the lowest number, in that base, into which all three will divide exactly.

Operations Defined on the Natural Numbers

Besides the four basic operations are included raising to a power and extracting a root. Raising to a power is an extension of multiplication, and index notation is used. Thus 'seventeen raised to the power five' means $17 \times 17 \times 17 \times 17 \times 17$. It is called seventeen to the fifth, seventeen is the base and five is the index.

Extracting a root is the reverse process and the symbol used is $\sqrt{}$.

Thus $2^3 = 8$ and $2 = \sqrt[3]{8}$
$3^4 = 81$ and $3 = \sqrt[4]{81}$
$5^2 = 25$ and $5 = \sqrt{25}$

By convention the 2 is usually omitted before a square root.

When a number is written as a root it is called a surd. $\sqrt{4}$, $\sqrt{25}$, $\sqrt[3]{27}$, $\sqrt[4]{81}$ are all surds.

1.3 The Laws for Operations

There are three laws, the commutative, associative and distributive laws.

The Commutative Law

An operation $*$ is said to be commutative if $a * b = b * a$ for every pair of numbers a, b, on which the operation is defined.

It is well known from elementary arithmetic that the operations of addition and multiplication are commutative for the natural numbers.

For example $7 + 11 = 11 + 7 = 18$
and $3 \times 4 = 4 \times 3 = 12$

However, subtraction and division are *not* commutative

$5 - 2 \neq 2 - 5$ ($\neq$ means 'is not equal to')
$6 \div 3 \neq 3 \div 6$

It is sufficient to find just one pair which does not commute to show that an operation is not commutative for certain numbers.

The Associative Law

An operation $*$ is said to be associative if $a*(b*c) = (a*b)*c$ for all numbers a, b, c. The operation is performed first on the pair in parentheses (brackets). For the natural numbers addition and multiplication are associative but subtraction and division are not.

For example $7 + (4 + 2) = (7 + 4) + 2 = 13$
and $7 \times (4 \times 2) = (7 \times 4) \times 2 = 56$
but $7 - (4 - 2) = 5$ and $(7 - 4) - 2 = 1$
$12 \div (6 \div 2) = 4$ and $(12 \div 6) \div 2 = 1$

The Distributive Law

This fixes the protocol for operations. Multiplication is distributive over addition and subtraction since for any three numbers represented by a, b, and c

$a \times (b + c) = (a \times b) + (a \times c)$
$a \times (b - c) = (a \times b) - (a \times c)$

Going from left to right, this is called 'removing the brackets', and going from right to left is it 'taking out a common factor'. Both are very important in algebra, and will be met again in chapter 4.

$3 \times (7 + 5) = 3 \times 12 = 36$
and $3 \times 7 + 3 \times 5 = 21 + 15 = 36$

Addition is *not* distributive over multiplication.

$7 + (3 \times 5) = 7 + 15 = 22$
but $7 + 3 \times 7 + 5 = 10 \times 12 = 120$

Because the distributive law holds for some pairs of operations and not others, it is important that calculations involving more than one operation should be performed in the correct order, Brackets, Multiplication and Division, Addition and Subtraction.

Example 1.18

Calculate
(a) $134 - 17 \times 4 + 2$
(b) $(134 - 17) \times (4 + 2)$

(c) $134 \times 7 + 134 \times 3$
(d) $120 \div (6 + 2) \times 16$

(a) The multiplication is performed first

$$\begin{aligned} & 134 - 17 \times 4 + 2 \\ = {} & 134 - 68 + 2 \\ = {} & \underline{68} \end{aligned}$$

(b) The brackets must be evaluated first

$$\begin{aligned} & (134 - 17) \times (4 + 2) \\ = {} & 117 \times 6 \\ = {} & \underline{702} \end{aligned}$$

(c) The distributive law can be used to shorten the calculations

$$\begin{aligned} & 134 \times 7 + 134 \times 3 \\ = {} & 134 \times (7 + 3) \\ = {} & 134 \times 10 \\ = {} & \underline{1340} \end{aligned}$$

(d) The addition must be done first, because of the bracket

$$\begin{aligned} & 120 \div (6 + 2) \times 16 \\ = {} & 120 \div 8 \times 16 \\ = {} & 15 \times 16 \\ = {} & \underline{240} \end{aligned}$$

Further examples on the use of these three laws are given in chapter 5 (operations on matrices), in chapter 7 (operations on sets), and in chapter 6 of volume 2 (binary operations).

Example 1.19

(a) Find a value of n for which 1400_n is a perfect square.
(b) Show that $(11_n)^2 = 121_n$ for all values of n greater than 2, and find the value of 11×11 in base 2.

(a) In any base $10 \times 10 = 100$
and $(A \times 10)^2 = A^2 \times 100$

Comparing $A^2 \times 100$ with 1400, we require a base in which 14 is a perfect square, and it must be at least 5. In base 5, $3 \times 3 = 14$. Therefore in base 5, $30 \times 30 = 1400$, $\underline{n = 5.}$

(b) In base 3, $11 \times 11 = 121$ and since 3 is the lowest base in which the digit 2 occurs, $11 \times 11 = 121$ in every base higher than 3.

$$\begin{array}{r} 11 \\ \times\ 11 \\ \hline 110 \\ 11 \\ \hline 121 \end{array}$$

$\underline{(11_n)^2 = 121_n \text{ for } n \text{ greater than } 2}$

In base 2

$$\begin{array}{r} 11 \\ \times\ 11 \\ \hline 110 \\ 11 \\ \hline 1001 \end{array}$$

$(11_2)^2 = \underline{1001}$

Exercise 1.5

A

Give the answer as a natural number

(1) $\sqrt{9}$ (2) $\sqrt{36}$ (3) $\sqrt{144}$ (4) $\sqrt{225}$.

Evaluate

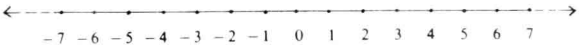
(5) $8 - (3 + 2)$	(6) $12 + (6 - 2)$
(7) $20 \times 12 + 1$	(8) $28 - 6 \times 4$
(9) $30 \div 2 + 3$	(10) $13 - 5 - 1$
(11) $22 \times (6 - 2)$	(12) $23 + 6 - 4 \times 2$
(13) $32 - 6 \div 2$	(14) $3 \times 4 \div 2 + 1$
(15) $25 - 6 \times 2$	(16) $43 - 3 \times (6 + 4)$.

B

(1) Find the denary equivalent of the highest three digit octal number.
(2) What is the lowest value of n that makes the given number a perfect square?
(a) 1100_n (b) 224_n (c) 213_n.
(3) State the base n in which these calculations are correct
(a) $313_n - 121_n = 142_n$ (b) $31_n \times 15_n = 525_n$.
(4) When 535_n is multiplied by 5_n the units digit is 1. Find the two lowest values of n.

1.4 Integers and Directed Numbers

The sum of any two natural numbers is itself a natural number, and so for the operation of addition the natural numbers are sufficient. Since a natural number can be subtracted only from one which is greater, to ensure that the difference of two numbers always has a meaning, the counting numbers must be extended to include zero and the negative whole numbers. The positive and negative whole numbers together with zero form the set of numbers called INTEGERS. Each integer can be represented by a point on a straight line which is assumed to extend indefinitely in both directions.



The numbers on the right of the zero position are the positive integers which are identical with the natural numbers, and to each natural number there corresponds a negative number on the left of the zero position.

The zero is defined as the integer 0 such that for every natural number n, $n + 0 = n$. The negative integer $-n$ is defined such that $-n + n = 0$.

For example, $-3 + 3 = 0$
and $6 - 6 = 0$.

The zero 0 is called the identity element for addition, and every negative integer is the additive inverse of a positive integer. More examples of identity and inverse elements will be met in chapter 6 (volume 2).

Order Relations defined on the Integers

The integers have a definite order of magnitude, and when represented on the number line an integer is greater than any integer further to the left and less than any integer further to the right.

The symbol $<$ is used to mean 'is less than' and the symbol $>$ 'is greater than'.

Thus $3 < 5$, $7 > 4$, $-2 > -5$

$\nless$ means 'is not less than', $\ngtr$ means 'is not greater than'

$\leqslant$ means 'is less than or equal to', $\geqslant$ means 'is greater than or equal to'.

If a number n is not less than another number m, it must be greater or equal, so the statements $n \nless m$, $n \geqslant m$ are equivalent. Similarly $m \ngtr n$, $m \leqslant n$ are equivalent.

The symbols for 'equal' $=$ and 'not equal' $\neq$ are already familiar, $<$ and $>$ are called inequality signs. Inequalities are studied further in chapter 8 and chapter 1 of volume 2.

Example 1.20

(a) Arrange in ascending order using the symbol $<$ the numbers 3, -17, -12, -5.

(b) Arrange in descending order, using the symbol $>$ the numbers 211_3, 121_4, 112_5.

(c) Arrange the following integers in order of magnitude -1111_2, -212_3, -15_6, -9.

(a) $-17 < -12 < -5 < 3$

(b) Changing to base 10

$211_3 = 18+3+1 = 22$

$121_4 = 16+8+1 = 25$

$112_5 = 25+5+2 = 32$

Therefore in descending order,

$112_5 > 121_4 > 211_3$.

(c) Expressed in base 10 the numbers are -15, -23, -11, -9. In ascending order,

$-212_3 < -1111_2 < -15_6 < -9$.

Operations defined on the Integers

Directed Numbers

The symbols for addition $+$ and subtraction $-$ are the same as the symbols for positive and negative numbers, and so to avoid confusion at first a signed (directed) number may be written with a prefix, such as $^-5$ meaning the negative integer -5, or $^+3$ meaning the positive integer 3.

Addition of Integers with the Same Sign

To add two positive or two negative integers, the numerical sum is calculated, ignoring the sign, and the result is given the common sign. For example

$^+3 + {}^+5 = {}^+8$

$^+2 + {}^+7 = {}^+9$

$^-5 + {}^-7 = {}^-12$

$^-2 + {}^-5 = {}^-7$

Since positive integers are not distinguished from natural numbers the positive signs are usually omitted.

Addition of Integers with Different Signs

To add one positive and one negative integer, the numerical difference of the unsigned numbers is calculated, and the result is given the sign of the numerically greater number. For example

$^-5 + 3 = {}^-2$

$5 + {}^-3 = 2$

Subtraction

Subtracting a positive integer is equivalent to adding the corresponding negative integer. For example

$3 - 3 = 3 + {}^-3 = 0$

$6 - 3 = 6 + {}^-3 = 3$

Subtracting a negative integer is equivalent to adding the corresponding positive integer. Thus

$6 - {}^-3 = 6 + 3 = 9$

$^-6 - {}^-3 = {}^-6 + 3 = {}^-3$

Example 1.21

$8 - 2 = 8 + {}^-2 = 6$

$^-8 - 2 = {}^-8 + {}^-2 = {}^-10$

$8 - {}^-2 = 8 + 2 = 10$

$^-8 - {}^-2 = {}^-8 + 2 = {}^-6$

The rule for subtracting an integer may be remembered as '*change the sign and then add*'.

Multiplication and Division of Integers

The product or quotient of two integers is defined as positive when they have the same sign and negative when they have different signs. For example

$3 \times 5 = 15$	$8 \div {}^-2 = {}^-4$
$^-3 \times {}^-5 = 15$	$8 \div 2 = 4$
$^-3 \times 5 = {}^-15$	$^-8 \div 2 = {}^-4$
$3 \times {}^-5 = {}^-15$	$^-8 \div {}^-2 = 4$

This may be remembered as *like signs* +, *unlike signs* −.

The product of any number with zero is zero, and division by zero has no meaning.

The logical justification for these rules is outside the scope of this book, but there are numerous mathematical models for verifying them, such as directed movement along the number line, and debit and credit of a bank balance.

Calculations with Integers in Different Number Bases

Just as with the counting numbers, the arithmetic of integers is the same whatever base is used to express them.

Exercise 1.6

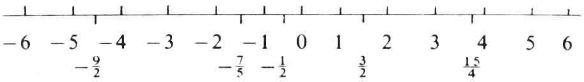
(1) Write down the numbers in ascending order using the symbol <
(a) 0, 4 (b) 0, $^-1$ (c) $^-2$, $^-4$ (d) 3, $^-5$, $^-1$
(e) 1, 0, $^-1$.
(2) Evaluate
(a) $^+3 - {}^+5$ (b) $^-2 + {}^-1$ (c) $^+3 - {}^+7$
(d) $^-1 - {}^-4$ (e) $^-2 + {}^-4$ (f) $^+4 - {}^-3$.
(3) Find the value of
(a) $^-3 \times {}^+5$ (b) $^+2 \times {}^-3$ (c) $^-2 \times {}^-4$
(d) $^-6 \div {}^+2$ (e) $4 \div {}^-1$ (f) $^-6 \div {}^-2$
(g) $^-3 \times {}^-2 \div {}^-1$ (h) $^+4 \div {}^-2 \times {}^+3$.
(4) Express the following as the product of prime factors in the given base
(a) 1010_2 (b) 110_2 (c) 1231_4 (d) 1100_3.
(5) If $a = 1101_2$, $b = 112_3$, $c = 14_5$ and $d = 1001_2$ which of the following are correct?
(i) $a > b$ (ii) $b > c$ (iii) $c = d$
(iv) $a < b$ and $b > c$ (v) $c = d$ and $b < c$.

1.5 Rational Numbers

Although the set of integers is sufficient for finding the sum, difference, and product of two integers, the number system must be extended yet again to cover division, since the quotient of two integers is itself an integer only if the division is by a factor.

A *rational* number is defined as one that can be expressed as the ratio (or quotient) of two integers. Thus 1/3, 2/5, 13/7, $^-15/3$, 1/2, 2/1 are all rational numbers, but only $^-15/3$ and 2/1 are integers. $^-15/3$ means $^-15 \div 3$ and is the integer $^-5$. 2/5 means $2 \div 5$, and is called two-fifths, or the ratio of 2 to 5.

The rational numbers which are not integers occupy positions between integers on the number line.



They are an ordered set, and an infinite number of rational numbers can be inserted between any two integers.

For example, between 0 and 1 we can insert 1/2, 2/3, 3/4, 4/5, 5/6, 6/7, 7/8, 8/9, 9/10, . . . 99/100, . . . 999/1000, . . .
These are in increasing order since for any integer n $n/(n+1)$ is always less than $(n+1)/(n+2)$ (proved in chapter 1 of volume 2) so that when $n = 1$, $1/2 < 2/3$ and when $n = 2$, $2/3 < 3/4$.

However, since $n < n+1$, the value 1 is never reached however far the series is extended. Similarly, between 3 and 4, we can insert $3\frac{1}{2}$, $3\frac{2}{3}$, etc. by adding 3 to each of the rational numbers listed above.

Each rational number can be written in many ways. Since $8 \div 4 = 2$ and $6 \div 3 = 2$, $8/4 = 6/3 = 2/1$, and the rational number 2 can be obtained by dividing an even integer by the corresponding integer which is half as great.

Similarly $\frac{15}{3} = \frac{30}{6} = \frac{60}{12}$ etc.

and $\frac{3}{15} = \frac{6}{30} = \frac{12}{60} = \frac{1}{5}$

Decimal Fractions

Rational numbers written in the denary system are called decimal fractions (or shortened to decimals). It was shown earlier in the chapter that dividing a number in base 10 by a multiple of 10 has the effect of moving each digit further to the right. The rational numbers extend in columns to the right of the units column.

The whole numbers are separated from the fractions by a dot . called a decimal point. Since $1 \div 10 = 1/10$, the first column to the right of the decimal point is headed 1/10, or tenths. But moving a digit to the right decreases the index number, or power of ten, so that $1/10 = 10^{-1}$, $1/10^2 = 10^{-2}$

. . . 10^3	10^2 hundreds	10^1 tens	10^0 units	.	10^{-1} tenths	10^{-2} hundredths	10^{-3} . . .
		2	5	.	7		
			1	.	0	9	
			0	.	1	5	

$$25.7 = 2 \times 10 + 5 \times 1 + 7 \times \frac{1}{10}$$